2021

MATHEMATICS

PAPER-I

Time Allowed — 3 Hours | *Full Marks — 200*

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

Answers may be given either in ***English*** *or in* ***Bengali*** *but all answers must be in one and the same language.*

1. Answer *any two* questions: $10 \times 2 = 20$

(a) If $U(F)$ and $V(F)$ are two vector spaces and T_1, T_2 are linear transformations from U into V, prove that the mapping T defined by $T(\alpha) = CT_1(\alpha) + T_2(\alpha), \alpha \in U, C \in F$ is a linear transformation from U into V.

(b) Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

(c) Reduce the matrix A to row-reduced echelon form and hence find its rank, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

2. Answer *any two* questions: $10 \times 2 = 20$

(a) Let $0 < x_0 < 1$ and $x_{n+1} = \frac{1}{1 + x_n}$ for all $n \geq 0$. Prove that $\{x_n\}_n$ converges.

(b) If $\underset{x \to 0}{\text{Lt}} \frac{\sin 2x + a \sin x}{x^3}$ be finite, find the value of 'a' and the limit. $a = -2, -\frac{1}{3}$

(c) Discuss the applicability of the Mean Value Theorem $f(b) - f(a) = (b - a)f'(\xi), a < \xi < b$. Also find ξ, where $f(x) = x(x-1)(x-3), 0 \leq x \leq 4$.

$\xi = \frac{8}{3}$

3. Answer *any two* questions: $10 \times 2 = 20$

(a) Let $f : S \to R, (S \subset R)$, be continuous on S and S be a compact set. Then prove that $f(S)$ is a compact set.

(b) Show that $\int_1^\infty \frac{\sin x}{x^p} dx$ converges for $p > 0$.

(c) Develop $f(x)$ in Fourier series on $-\pi < x < \pi$ if

$$f(x) = 0, \text{ for } -\pi < x < 0$$
$$= \pi, \text{ for } 0 < x < \pi.$$

10×2=20

4. Answer *any two* questions:

(a) A function $f(x)$ is defined on $[0, 1]$ by $f(x)=\begin{cases} x, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational.}\end{cases}$

Find the upper and lower integral sums corresponding to the partition P_n of $0 \le x \le 1$ into n equal partial intervals by points $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{r}{n}, \cdots, \frac{n}{n}\right\}$; hence evaluate I and J and show that $f(x)$ is not integrable on $[0, 1]$.

(b) Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$ is not uniformly convergent on $[0, 1]$.

$2\sin x\cos x + 2\cos^2 x + 1 -$

(c) If $y = 2\cos x(\sin x - \cos x)$, show that $(y_{10})_0 = 2^{10}$.

$y = \sin 2x - 2\cos^2 x + 1 - 1$
$= \sin 2x - 2\left(\frac{1-\cos 2x}{2}\right)$

5. Answer *any two* questions: 10×2=20

(a) If the line $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonal, show that the equation of the other diagonal is $y(bl - hm) = x(am - hl)$.

(b) If the normal to the hyperbola $xy = c^2$ at the point $\left(ct_1, \frac{c}{t_1}\right)$ meets the curve again at the point $\left(ct_2, \frac{c}{t_2}\right)$, then show that $t_1^3 t_2 + 1 = 0$.

(c) Reduce the equation $x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$ to its canonical form and determine the nature of the conic.

6. Answer *any two* questions: 10×2=20

(a) Show that the locus of a point which is equidistant from two given straight lines $y = mx, z = c$ and $y = -mx, z = -c$ is $mxy + c(1 + m^2)z = 0$.

(b) Show that the feet of the normals from the point (α, β, γ) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie on the surface $\frac{\alpha a^2(b^2 - c^2)}{x} + \frac{\beta b^2(c^2 - a^2)}{y} + \frac{\gamma c^2(a^2 - b^2)}{z} = 0$.

(c) Reduce the equation $x^2 + y^2 + z^2 - 2yz + 2zx - 2yx + x - 4y + z + 1 = 0$, to its canonical form and determine the type of the quadric represented by it.

7. Answer *any two* questions: 10×2=20

(a) Solve : $(x - y^2)\,dx + 2xy\,dy = 0$ $\quad \log x + \frac{y^2}{x} = c$

(b) Solve : $(D^2 - 3D + 2)\,y = \cos 3x$

(c) Find the orthogonal trajectory of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$.

$y = \sin 2x - 2\cos^2 x$
$= \sin 2x - (2\cos^2 x - 1) - 1$
$= \sin 2x - \cos 2x - 1$

8. Answer *any two* questions: $10\times2=20$

(a) Solve : $\sin x \dfrac{d^2y}{dx^2} - \cos x \dfrac{dy}{dx} + 2y\sin x = 0$

(b) Solve by using Laplace transform, the equations $(D-2)x+3y=0,\ 2x+(D-1)y=0,\ t>0$ and $D=\dfrac{d}{dt}$ given that $x(0)=8$ and $y(0)=3$.

(c) Solve : $(x^2-yz)\,p+(y^2-zx)\,q=(z^2-xy)$, where $p=\dfrac{\partial z}{\partial x}$ and $q=\dfrac{\partial z}{\partial y}$.

9. Answer *any two* questions: $10\times2=20$

(a) Forces X, Y, Z act along the three straight lines $y=b,\ z=-c;\ z=c,\ x=-a;\ x=a,\ y=-b$ respectively. Show that they will have a single resultant at $\dfrac{a}{X}+\dfrac{b}{Y}+\dfrac{c}{Z}=0$.

(b) A hemispherical shell on a rough plane, whose angle of friction is λ, show that the inclination of the plane base of the rim to the horizontal cannot be greater than $\sin^{-1}(2\sin\lambda)$.

(c) The moments of a system of forces about the points $(0, 0), (a, 0), (0, a)$ are $a\omega, 2a\omega, 3a\omega$ respectively. Find the components of their resultant parallel to the coordinate axes and the equation to its line of action.

10. Answer *any two* questions: $10\times2=20$

(a) A particle moves from rest in a straight line under an attractive force $\mu\times(\text{distance})^{-2}$ per unit mass to a fixed point on the line. Show that if the initial distance from the centre of force be $2a$, then the distance will be a after a time $\left(\dfrac{\pi}{2}+1\right)\left(\dfrac{a^3}{\mu}\right)^{1/2}$.

(b) A body is projected horizontally from a point on the earth's surface with velocity $\sqrt{1\cdot5Rg}$, R being the earth's radius. What will be its maximum distance from the earth's centre?

(c) A particle describes the equiangular spiral $r=ae^{\theta}$ in such a manner that the radial acceleration is zero. Prove that the speed and the magnitude of acceleration are each proportional to r.

$\dfrac{d^2x}{dt^2}=\dfrac{\mu}{x^2}$ $\quad \dfrac{dx}{dt}=$ $\quad x^2\dfrac{d^2x}{dt^2}-4=0$

$\dfrac{x-0}{0}=\dfrac{y-6}{0}=$

$x=e^z$

$D(D-1)-1=0$

$D^2-D-1=0$

$\dfrac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}=\dfrac{d(x-y)}{(x-y)(x+y+z)}$

$\dfrac{dx-dy}{(x+y)(x-y)+z(x-y)}=\dfrac{d(x-y)}{(x-y)(x+y+z)}=\dfrac{d(y-z)}{(y+z)(y-z)+x(y-z)}=\dfrac{d(y-z)}{(y-z)(x-}$

$m^2-m-1=0$

$m=\dfrac{1\pm\sqrt{1+4}}{2}=\dfrac{1\pm\sqrt5}{2}$

$\dfrac{d(x+y+z)}{(x^2+y^2+z^2-xy-yz-zx)}=\dfrac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-\dots)}$

$\dfrac{1+\sqrt5}{2}$.

$3x-8y+4=0$