

2019

MATHEMATICS

PAPER-I

Time Allowed: 3 Hours

Full Marks: 200

If the questions attempted are in excess of the prescribed number, only the questions attempted first up to the prescribed number shall be valued and the remaining ones ignored.

Answer may be given either in English or in Bengali but all answer must be in one and the same language.

1. Answer any two questions:

10×2=20

(a) Prove that $S = \{(x, y, z, w) \in \mathbb{R}^4; 2x + y + 3z + w = 0, x + 2y + z + 3w = 0\}$ is a vector space. Find its dimension and a basis.

(b) A is a non-singular square matrix of order 3 such that sum of the element of each row is K . Hence show that $(1, 1, 1)^T$ is an eigenvector of A . Hence show that sum of the elements in each row is $\frac{1}{K}$.

(c) If $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, show that 2 is an eigenvalue of A . Find its algebraic and geometric multiplicity. Hence comment on diagonalisability of A .

2. Answer any two questions:

10×2=20

(a) If the system of equations $ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0$ has a non-zero solution, then prove that either $a = b = c$ or $a + b + c = 0$.

(b) Find the nature of the quadratic form $Q = xy + yz + zx$ and obtain a non-singular transformation which reduce it into normal form.

(c) If S be a skew Hermitian matrix then show that $(I + S)(I - S)^{-1}$ is a unitary matrix.

3. Answer any two questions:

10×2=20

(a) Prove that between two real numbers there lie a rational and an irrational number.

(b) Prove that $\left\{\frac{x^n}{n}\right\}$ is converges to 0 $\forall x \in \mathbb{R}$.

(c) For a sequence $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, prove that $\{a_n\}$ is monotone increasing and divergent. Is $\sum \frac{1}{n}$ convergent? — Justify.

4. Answer any two questions:

10×2=20

(a) Show that $f: [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is N integrable but not R -integrable in $[0, 1]$.

(b) Show that the sequence of function $\{x^n\}, x \in [0, 1]$ is not uniformly convergent.

(c) Find interval of convergence of power series $\sum a_n x^n$ where $a_n = \frac{1}{(n+1)^2}$. Also comment on uniform convergence of this power series.

5. Answer any two questions:

10×2=20

(a) Using Lagrange Mean Value Theorem, prove that any chord of the parabola $y = ax^2 + bx + c$ is parallel to the tangent at the point whose abscissa is same as the middle point of the chord.

(b) If $f'(x)$ exist in $[0, 1]$ prove that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0, 1)$.

(c) Find the envelop of the circles drawn upon the radius vectors of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameter.

6. Answer any two questions:

10×2=20

(a) Reduce the equation $x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$ into canonical form and hence find the nature of the conic.

(b) Show that six normals can be drawn from a point to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(c) Prove that locus of the point of intersection of the normals to the parabola $y^2 = 4ax$ at the extremities of the focal chord is the parabola $y^2 = a(x - 3a)$.

7. Answer any two questions:

10×2=20

(a) Show that the lines $\frac{x-a_1}{a_2} = \frac{y-b_1}{b_2} = \frac{z-c_1}{c_2}$ and $\frac{x-a_2}{a_1} = \frac{y-b_2}{b_1} = \frac{z-c_2}{c_1}$ will intersect. Find their point of intersection.

(b) The points $(0, 1, 0)$ and $(3, -5, 2)$ are end points of a diameter of a sphere S . Find equation of the sphere on which intersection of the plane $5x - 2y + 4z + 7 = 0$ with the given sphere S is a great circle.

(c) Find the integrating factor of

$$x(3ydx + 2xdy) + 8y^4(ydx + 3xdy) = 0 \text{ of the form } x^\alpha y^\beta \text{ and hence solve it.}$$

8. Answer any two questions:

10×2=20

(a) Find general and singular solution of

$$p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0, \text{ where } p = \frac{dy}{dx}.$$

✓ (b) Solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cosh x$

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(c) Solve by Charpit method: $(p^2 + q^2)y = qz$.

9. Answer any two questions:

10×2=20

(a) Find the condition of astatic equilibrium in 2-dimension.

(b) A paraboloid of revolution is fixed with its axis vertical and vertex upward, a heavy elastic string of unstretched length $2\pi c$ is placed on it. Show that in equilibrium it rests in form of a circle of radius $\frac{4\pi ac}{4\pi a\lambda - cw}$; where w is the weight of the string, λ is modulus of elasticity and $4a$ is the latus rectum of the generating parabola.

(c) Forces X, Y, Z act along three lines $y = b, z = -c; z = c, x = -a; x = a, y = -b$ respectively show that they will have a single resultant if $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$.

10. Answer any two questions:

10×2=20

(a) A particle moves in a plane under a force which is always perpendicular and towards a fixed straight line on the plane, magnitude being $\mu \div (\text{distance from the line})^2$. If initially it be at a distance $2a$ from the line and projected with a velocity $\sqrt{\frac{\mu}{a}}$ parallel to the line, prove that the path traces out by it is a cycloid.

(b) A particle describes the path $r^4 = a^4 \cos 4\theta$ under a central force. Find the law of force.

(c) State and prove Kepler's Second law on planetary motion.