CSM – 52/21
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and any ***three*** *of the remaining questions, selecting at least* ***one*** *from each Section.*

SECTION – A

1. (a) Let N and M be normal subgroups of a group G such that $N \cap M = \{e\}$, where e is the identity element of group G. Show that $nm = mn$ for all $n \in N$ and $m \in M$. 15

(b) Is the vector $(2, -5, 3)$ in a subspace of $\mathbb{R}^3$ spanned by the vectors $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$? 15

(c) Show that $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$ represents lines forming a parallelogram and find the length o: the sides. 15

(d) Find the symmetric form of the equations of the line $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$. 15

2\. (a) Prove that a group G is abelian if and only if the mapping $f : G \to G$ given by $f(x) = x^{-1}$, is a homomorphism. 15

(b) Let $A = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $B = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^3$. Find the dimension of $A + B$. 15

(c) Show that every square matrix A with entries are from the set of complex numbers, can be uniquely written as $P + iQ$, where P and Q are Hermitian matrices. 15

(d) Obtain the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 6x + 4y - 3 = 0$, passes through the point (3, 0) and touches the axis of y. 15

3. (a) Let $n > 1$ be a fixed and a, b, c, d be arbitrary integers. If $ac \equiv bc \pmod{n}$ then show that $a \equiv b \pmod{n/d}$, where $d = \gcd(c, n)$. 15

(b) Let H be a subgroup of permutation group S_n, $n \geq 2$. If H contains an odd permutation, prove that the set of all even permutations in H forms a normal sub-group of H of index 2. 15

(c) Find the equation of the right circular cylinder through the circle of intersection of sphere $x^2 + y^2 + z^2 = 1$ and plane $x + y + z = 1$. 15

(d) Show that ring $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}$ and $i = \sqrt{-1}\}$ of Gaussian integers is an Integral Domaim. Determine the units of $\mathbb{Z}[i]$. 15

4. (a) Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be eigenvalues of a square matrix A. Determine the eigenvalues of A^2 and $C^{-1}AC$, where C is any inversible matrix. 15

(b) Find the equations of the spheres through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touching the plane $z = 0$. 15

(c) Determine the rank and nullity of linear transformation $T : \mathbb{R}^3(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ defined by $T(x, y, z) = (x + y, z)$ for all $(x, y, z) \in \mathbb{R}^3$. 15

(d) Show that the set of nilpotent elements of a ring R is an ideal of R. 15

SECTION – B

5. (a) Prove that the series $\sum_{k=1}^{\infty} \frac{1}{k^2 + k}$ converges and find its sum. 15

(b) Find the volume of the solid formed by revolving the region bounded by the x-axix and the graphs of $y = x^3 + x^2 + 1$, $x = 1$ and $x = 3$ about the y-axis. 15

(c) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be mutually perpendicular vectors of equal magnitute. Show that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. 15

(d) Determine the harmonic conjugate of the function $u(x, y) = x^3 - 3xy^2 - 5y$. 15

6. (a) Show that the function $f(x) = |x| - 1$, for all $x \in \mathbb{R}$, is derivable at all points except $x = 0$. 15

(b) Find constants a, b, c so that the vector $\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy - 2z)\hat{k}$ is irrotational. 15

(c) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$. 15

(d) Find the integral of $f(x, y) = x^4 + y^2$ over the region bounded by $x = y^{1/3}$ and $x = \sqrt{y}$. 15

7. (a) Discuss the singularities of $f(z) = \frac{z-1-i}{z^2-(4+3i)z+(1+5i)}$. 15

(b) Find the value of the integral $\int_S \vec{F} \cdot \hat{n} \, dS$, where $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ is the unit outward drawn normal vector to the surface S. 15

(c) Show that the integral $\int_0^{\pi/2} \left(\frac{\sin^m x}{x^n} \right) dx$ exists if and only if $n < m + 1$. 15

(d) Determine the greatest lower bound of the set $A = \{4 + n^2 + \frac{1}{n^2} \mid n \in \mathbb{N}\}$. 15

8. (a) Show that the sequence $< f_n(x) >$ of functions, where $f_n(x) = nxe^{-nx^2}$, $x \geq 0$, is not uniformly convergent on $[0, k]$, $k > 0$. 15

(b) Using Green's theorem, evaluate the integral $\oint_C (xy + y^2)\ dx + x^2\ dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. 15

(c) Write Laurent series expansions of

$$f(z) = \frac{1}{z(z-1)}$$ for the domains : (a) $0 < |z| < 1$

and (b) $1 < |z|$. 15

(d) Find the area included between the curve $x^2y^2 = 4(y^2 - x^2)$ and its asymptotes. 15