CSM – 52/20
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

*Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and any **three** of the remaining questions, selecting at least **one** from each Section.*

SECTION – A

1. (a) Prove that the order of each element in a finite group G is a divisor of O(G). Also prove that for any $a \in G$, $a^{O(G)} = e$, e is the identity element of G. 15

(b) Find the dimension of the subspace W of $\mathbb{R}^3$ defined by $W = \{(x, y, z) : x, y, z \in \mathbb{R}$, $x - y + z = 0, 2x + y - z = 0$. 15

(c) Find the angle between the lines joining the origin to the intersection of the line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$. 15

(d) Find the equation of the cone whose vertex is $(1, 0, -1)$ and which passes through the circle $x^2 + y^2 + z^2 = 4, x + y + z = 1$. 15

2. (a) Prove that the ring $\mathbb{Z}[x]$, the ring of all polynomials with integer coefficients, is an integral domain. 15

(b) Check whether the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}$

is unitary or not.

15

(c) Let V be a vector space of 2×2 matrices over $\mathbb{R}$. Let $T : V \to V$ be a linear transformation defined by :

$T(A) = AM - MA$ where

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

Find the basis and dimension of Ker T. 15

(d) If two circles cut a third circle orthogonally, show that the centre of the third circle lies on the radial axis of the two circles. 15

3. (a) If α is an eigen value of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is an eigen value of adj A ($|A|$ is the determinant of A). 15

(b) In a ring $R = \mathbb{Z}$, prove that every prime ideal is a maximal ideal. 15

(c) Let H be a cyclic subgroup of a group G. If H is normal in G, prove that every subgroup of H is normal in G. 15

(d) Show that the vectors (1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1) and (0, 0, 0, 1) is a basis of $\mathbb{R}^4$ over $\mathbb{R}$. 15

4. (a) Solve the congruence equation $8x \equiv 12 \pmod{28}$. 15

(b) Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ intersect in a line lying in the plane $x = 0$. Show that the line touches the parabola $x = 0$, $y^2 = (a + b)(2z + a)$. 15

(c) Find the eigen values of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and also prove that $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$. 15

(d) Show that the integral domain $\mathbb{Z}\left[\sqrt{-5}\right]$ is not a unique factorization domain. 15

SECTION – B

5. (a) Let $0 < x_1 < x_2$ and for the sequence $\{x_n\}$ we have $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ $(n \geq 3)$, then prove that $\{x_n\}$ converges to $\dfrac{x_1 + 2x_2}{3}$. 15

(b) A parametric curve defined by $x = \cos\left(\dfrac{\pi t}{2}\right)$, $y = \sin\left(\dfrac{\pi t}{2}\right)$ $(0 \leq t \leq 1)$ is rotated about x-axis by 360°. Find the area of the surface generated. 15

(c) Find the maximum value of the directional derivative of the function $f(x, y, z) = xy + yz + zx$ at the point $P(-1, 1, 1)$. 15

(d) Find the analytic function of a complex variable $z = x + iy$ whose real part is $2xy$. 15

6. (a) Find the integral $\frac{1}{2\pi}\iint_D (x+y+10)\,dx\,dy$, where D denotes the dice : $x^2 + y^2 \le 4$. 15

(b) Find the nature of singularity of $f(z) = \cot z$ at $z = \infty$. 15

(c) A function f assume only rational values in [0, 1] but continuous in this interval. If $f\left(\frac{1}{2}\right) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ every where in [0, 1]. 15

(d) Prove that the asymptotes of the curve $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ form a triangle of area a^2. 15

7. (a) Given a vector $\overrightarrow{u} = \frac{1}{3}(-y^3\hat{i} + x^3\hat{j} + z^3\hat{k})$ and $\hat{n}$ as a unit normal to the surface of the hemisphere $(x^2 + y^2 + z^2 = 1, z \geq 0)$. Find the value of the integral $\int(\nabla \times \overrightarrow{u}) \cdot \hat{n}ds$ evaluated on the curved surface of the hemisphere S. 15

(b) If $u = 2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \cot\frac{u}{2} = 0$. 15

(c) Let $f(x) = 2x^3 - x^4 - 10$ be a real function defined in $-1 \leq x \leq 1$. Then find the minimum value of f(x) in [–1, 1]. 15

(d) Prove that the integral $\int_1^\infty \frac{x^2dx}{(1+x^2)^2}$ is convergent and hence find its value. 15

8. (a) Find the length of the loop of the curve $3ay^2 = x(x-a)^2$. 15

(b) Prove that the vector $f(r)\overrightarrow{r}$ is irrotational $(\overrightarrow{r} = x\hat{i} + y\hat{j} + z\hat{k})$. 15

(c) Find the value of the integral

$$\int_c \frac{\cos 2\pi z}{(2z-1)(z-3)} dz, \text{ where c is } |z| = 1.$$ 15

(d) Show that the function defined by

$$f(x) = \frac{1}{2^n} \text{ for } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \quad n = 0, 1, 2........$$

and f(0) = 0 is integrable over the interval

[0, 1] and $\int_0^1 f(x)dx = \frac{2}{3}$. 15