CSM – 52/19
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and any ***three*** *of the remaining questions, selecting at least* ***one*** *from each Section.*

SECTION – A

1. (a) Let p be a prime and m, a positive integer such that p^m divides o(G). Then prove that there exists a subgroup H of G such that $o(G) = p^m$. 15

(b) Prove that if V is a finite dimensional vector space and $\{v_1, v_2, \ldots\ldots, v_n\}$ is a linearly independent subset of V, then it can be extended to form a basis of V. 15

(c) If H and K are finite subgroups of group G of order o(H) and o(K) respectively, then prove that $0(HK) = \frac{o(H)o(K)}{o(H \cap K)}$. 15

(d) Find the equation of the sphere circumscribing the tetrahedron bounded by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$ and find its radius and centre. 15

2. (a) Find all the homomorphisms from $Z/4Z$ to $Z/6Z$. 15

(b) Let R be a commutative ring with unity. Let A be an ideal of R. Show that $\frac{R[x]}{A[x]} \cong \frac{R}{A}[x]$.

Hence, prove or disprove that if A is prime ideal of R, then A[x] is prime ideal of R[x]. 15

(c) Let V and W are two vectors spaces and let $T : V \to W$ be a linear transformation, then Rank T + Nullity T = dim V. 15

(d) Find the enveloping cylinder of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the equations of whose generators are $x = y = z$. 15

3. (a) Let R be a ring having more than one element such that $aR = R$ for all $0 \neq a \in R$. Show that R is a division ring. 15

(b) Prove that all vectors in the vector space $\mathbb{R}^3$ with $v_2 - v_1 + 4v_3 = 0$ is a subspace of $\mathbb{R}^3$. Determine a basis and the dimension of the subspace. 15

(c) Diagonalize the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and find the modal matrix. Hence find A^4. 15

(d) Check whether the matrix $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is Hermitian or Skew-Hermitian or unitary. Find its eigenvalues and eigenvectors. 15

4. (a) Find the range, rank, kernel and nullity of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$. 15

(b) Show that every subgroup of an abelian group is normal. Give an example. 15

(c) The line of intersection of a pair of perpendicular tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, passes through a fixed point $(0, 0, \alpha)$ Show that the line of intersection lies on the cone $x^2(b^2 + c^2 - \alpha^2) + y^2(c^2 + a^2 - \alpha^2) + (z - \alpha)^2 (a^2 + b^2) = 0$. 15

(d) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. 15

SECTION – B

5. (a) Let $f : [0, 1] \to [0, 1]$ be a contraction map. Then (i) f has a unique fixed point $\ell \in (0, 1)$ and (ii) given $x_0 \in [0,1]$, there exists a sequence (x_n) defined by the iteration scheme $x_{n+1} = f(x_n)$; n = 0, 1, 2, ... such that $x_n \to \ell$ and $| x_n \to \ell |$

$$\leq \frac{c^n |x_1 - x_0|}{1-c}, n \geq 1.$$ 15

(b) The arc of the cardioids $r = a(1 + \cos\theta)$ included between $-\pi/2 \leq \theta \leq \pi/2$ is rotated about the line $\theta = \pi/2$. Show that the area of the surface thus generated is $48\sqrt{2\pi a^2}/5$. 15

(c) Find the equation of the tangent plane and the normal line to the surface $yz - zx + xy + 5 = 0$ at the point (1, –1, 2). 15

(d) Prove that a monotonic increasing sequence (x_n) bounded above is convergent and $\lim x_n = \sup x_n \in R$. Also, prove that the sequence $x_n = (1 + \frac{1}{n})^n$ converges. 15

6. (a) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) =$

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x < \frac{1}{n}, \text{ where } n \in N. \\ 0, & x = 0 \end{cases}$$

Show that f is integrable and

$$\int_0^1 f(x)dx = \frac{\pi^2}{6} - 1. \qquad 15$$

(b) If $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, show that the Cauchy-Riemann equation hold at $z_0 = x_0 + iy_0$. Again, show that the function f defined by $f(z) = |\operatorname{Re} z \operatorname{Im} z|^{1/2}$ satisfies the Cauchy-Riemann equation at origin. Is it differentiable at origin ? Justify. 15

(c) Show that the radius of curvature of the curve given by $x^2y = a\left(x^2 + \frac{a^2}{\sqrt{5}}\right)$ is the least for the point $x = a$ and its value there is $\frac{9a}{10}$. 15

(d) Show that the improper integral $I = \int_1^x \frac{\sin t}{t^p} dt$ is convergent if $p > 0$ and test the convergence of the integral $\int_1^x \sin\frac{1}{x^2} dx$. 15

7. (a) Evaluate $I = \oint_c \frac{z^2 + 4}{z^3 + 2z^2 + 2z} dz$ where c is $|z| = 1$. 15

(b) Show that the function $f(x, y) = \frac{e^{-|x-y|}}{x^2 - 2xy + y^2}$, when $(x, y) \neq (x, x)$ and $f(0, 0) = 0$ is continuous at $(0, 0)$. 15

(c) Verify Stoke's theorem for $F = [y^2, xy, -xz]$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. 15

(d) Find the volume of the solid surrounded by the surface $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$ 15

8. (a) Prove that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n \sin\left(\frac{m\pi}{n}\right)}$ for m, $n \in N$ with $n > m > 0$. 15

(b) Verify Green's theorem, find the area of the region in the first quadrant bounded by the curves $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$. 15

(c) Prove that the vector function $F = [6xy + z^3, 3x^2 - z, 3xz^2 - y]$ is irrotational. Find a scalar function $f(x, y, z)$ such that $F = \nabla f$. 15

(d) Find all possible Laurent series of $f(z) = \frac{7z^2 + 9z - 18}{z^3 - 9z}$ about its singular points. 15