CSM – 52/16
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

*Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and **three** of the remaining questions, selecting at least **one** from each Section.*

SECTION – A

1. Answer any **three** of the following :

   (a) If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that G / N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication. 20

(b) (i) Let V be the vector space of all 2 × 2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V. 12

(ii) Using Caley-Hamilton theorem, find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. 8

(c) A pair of tangents to the conic $ax^2 + by^2 = 1$ intercepts a constant distance 2k on the y-axis. Prove that the locus of their point of intersection is the conic : 20

$$ax^2(ax^2 + by^2 - 1) = bk^2(ax^2 - 1)^2$$

(d) Show that the length of the shortest distance between the line $z = x \tan\alpha$, $y = 0$ and any tangent to the ellipse $x^2\sin^2\alpha + y^2 = a^2$, $z = 0$ is constant. 20

(e) Find the equation of the sphere inscribed in the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 6z = 6$. 20

2. (a) Verify that the set E of the four roots of $x^4 - 1 = 0$ forms a multiplicative group. Also prove that a transformation T, $T(n) = i^n$ is a homomorphism from I_+ (Group of all integers with addition) onto E under multiplication. 15

(b) Prove that if the cancellation law holds for a ring R then $a (\neq 0) \in R$ is not a zero divisor and conversely. 15

(c) If p is a prime number of the form 4n + 1, n being a natural number, then show that congruence $x^2 \equiv -1 \mod p$ is solvable. 15

(d) If M and N are normal sub-groups of a group G such that $M \cap N = \{e\}$, show that every element of M commutes with every element of N. 15

3. (a) If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A = \begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix}$ then, show that

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \leq \sqrt{1949}.$$ 15

(b) Let V be the vector space of polynomials in x of degree $\leq$ n over $\mathbb{R}$. Prove that the set $\{1, x, x^2, ..., x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x. 15

(c) If S is a Skew-Hermitian matrix, then show that $A = (I + S)(I - S)^{-1}$ is a unitary matrix. Also, show that every unitary matrix can be expressed in the above fom provided -1 is not an eigen value of A. 15

(d) Investigate for what values of λ and μ the equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have : 15

(i) No solution

(ii) A unique solution

(iii) Infinitely many solutions

4. (a) Show that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines, if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. 15

(b) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$. 15

(c) Prove that the locus of the foot of the perpendicular drawn from the vertex on a tangent to the parabola $y^2 = 4ax$ is $(x + a)y^2 + x^3 = 0$. 15

(d) A line with direction ratios 2, 7, – 5 is drawn to intersect the lines $\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{4}$ and $\frac{x-11}{3} = \frac{y-5}{-1} = \frac{z}{1}$. Find the co-ordinates of the points of intersection and the length intercepted on it. 15

SECTION – B

5. Answer any **three** of the following :

(a) (i) For two vectors $\vec{a}$ and $\vec{b}$ given respectively by $\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{b} = \sin t\,\hat{i} - \cos t\,\hat{j}$, determine $\frac{d}{dt}\left(\vec{a} \cdot \vec{b}\right)$ and $\frac{d}{dt}\left(\vec{a} \times \vec{b}\right)$. 5+5 = 10

(ii) If u and v are two scalar fields and $\vec{f}$ is a vector field such that $u\vec{f} = \vec{\nabla} v$, find the value of $\vec{f} \cdot (\vec{\nabla} \times \vec{f})$. 10

(b) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{3 - i}{2}$. 20

(c) Let $S = (0, 1]$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \leq 1$ (in $\mathbb{R}$). Is f uniformly continuous on S ? Justify your answer. 20

(d) (i) Evaluate $\int_0^1 \log_e x \, dx$. 12

(ii) Evaluate $\lim_{x \to 2} f(x)$ 8

$$\text{where } f(x) = \frac{x^2 - 4}{x - 2} \quad x \neq 2$$
$$= \pi \quad x = 2$$

6. (a) If f is the derivative of some function defined on [a, b], prove that there exists a number $\eta \in [a, b]$, such that $\int_a^b f(t)dt = f(\eta)(b - a)$. 15

(b) Suppose that f'' is continuous on [1, 2] and that f has three zeros in the interval (1, 2). Show that f' has at least one zero in the interval (1, 2). 15

(c) Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$. Assume that the zeros of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\dfrac{a_{n-1}}{b_n}$. 15

(d) Show that $\vec{\nabla} \cdot (\vec{\nabla} r^n) = (n + 1)n\, r^{n-2}$, where $r = \sqrt{x^2 + y^2 + z^2}$. 15

7\. (a) Find the work done in moving the particle once round the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1, z = 0$ under the field of force given by $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$. 20

(b) Using divergence theorem evaluate $\iint_S \vec{A} \cdot d\vec{S}$ where $A = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. 20

(c) Find the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) . d\vec{s}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$ when $\vec{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$. 20

8. (a) Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ is not differentiable at $z = 0$. 15

(b) Using Lagrange's mean value theorem, show that $|\cos b - \cos a| \leq |b - a|$. 15

(c) A figure bounded by one arch of a cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$ and the x-axis is revolved about the x-axis. Find the volume of the solid of revolution. 15

(d) Show that $e^{-x}x^n$ is bounded on $[0, \infty)$ for all positive integral values of n. Using this result show that $\int_0^\infty e^{-x}x^n dx$ exists. 15