MATHEMATICS

Paper–I

Time Allowed : Three Hours

Full Marks : 300

*Candidates should attempt Question No. 1 from Section-A and Question No. 5 from Section-B which are compulsory, and any **three** of the remaining questions, selecting at least **one** from each Section.*

All questions carry equal marks.

SECTION—A

1. Attempt any **FIVE** of the following :—

(a) Prove that a subgroup H of a group G is normal in G, if and only if the following implication holds for any

$$a, b \in G : \quad ab \in H \Rightarrow ba \in H. \qquad 12$$

(b) If R is a commutative ring with identity, prove that R is a field if and only if R has no non-trivial ideals. 12

(c) Prove that every finite-dimensional vector space has a basis. 12

(d) Prove that a real n×n matrix A is orthogonal, if and only if the columns of A are mutually orthogonal unit vectors. 12

(e) If the limiting points of a system of coaxial circles are (1, 2) and (3, 5), find the equation of the member of the system which passes through the point (2, 2). 12

(f) Show that the plane $lx + my + nz = p$ will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$. 12

2. (a) If $a_1, a_2, \ldots\ldots, a_m$ is a complete residue system (mod m) and $(k, m) = 1$, then prove that $ka_1, ka_2, \ldots\ldots, ka_m$ is also a complete residue system (mod m). 15

(b) Prove that every finite group of order n is embeddable in the symmetric group S_n of degree n. 15

(c) Prove that the polynomial ring F[X] over a field F is a principal ideal ring. 15

(d) Prove that every group of order 15 is cyclic. 15

3. (a) Define the annihilator A(W) of a subspace W of a vector space V. If V is a finite-dimensional vector space, with usual identifications, prove that $W = A(A(W))$ for any subspace W of V. 15

(b) If U, W are subspaces of a vector space V of dimensions 5 and 7 respectively, and if $\dim(U \cap W) = 3$, find the $\dim(U + W)$. 15

(c) Show that, for $n > 1$, there do not exist two $n \times n$ real matrices A and B such that $AB - BA = I_n$, where I_n is the $n \times n$ identity matrix. 15

(d) Let f be any map from $\mathbb{R}^n$ to itself such that $\| f(x) - f(y) \| = \| x - y \|$ for all $x, y \in \mathbb{R}^n$. Then prove that there exists an orthogonal matrix A and a vector $c \in \mathbb{R}^n$ such that $f(x) = Ax + c$ for all $x \in \mathbb{R}^n$. 15

4. (a) If e and e' are the eccentricities of a hyperbola and its conjugate respectively, then prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$. 15

(b) Find the area of the triangle whose vertices are the points (1, 2, 3), (–2, 1, 0), (3, 4, –2). 15

(c) Find equations to the lines in which the plane $2x + y - z = 0$, cuts the cone $x^2 - y^2 + 3z^2 = 0$. 15

(d) Determine the nature of the section of the paraboloid $ax^2 + by^2 = 2cz$, by the plane $lx + my + nz = p$. 15

SECTION—B

5. Attempt any **FIVE** of the following :—

(a) Prove that the sequence $\{x_n\}$ where

$$x_1 = \sqrt{7},\ x_{n+1} = \sqrt{7 + x_n}$$

converges to the positive root of the equation $x^2 - x - 7 = 0$. 12

(b) Let $f(z)=\frac{z^2+5z+6}{z-2}$.

Can we apply Cauchy's integral theorem when

(i) the path C of the integral is a circle $|z|=3$?

(ii) the path C of the integral is a circle $|z|=1$?

Give reasons to support your answer. Evaluate the integral in each case. 12

(c) Find the points of minima for the function

$$f(x)=\int_{-1}^{x}+(e^t-1)(t-1)(t-2)^3(t-3)^5\,dt.$$

12

(d) Show that the integral

$$\int_0^\infty e^{-ax}\frac{\sin x}{x}dx, \quad a\geq 0$$

is convergent. 12

(e) Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors of magnitude 1, 1 and 2 respectively. If

$$\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=\vec{0},$$

then find acute angle between $\vec{a}$ and $\vec{c}$. 12

(f) Find the volume cut from a sphere of radius 'a' by a right circular cylinder with b as radius of the base and whose axis passes through the centre of the sphere. 12

6. (a) If f is Riemann integrable over the interval [a, b] then prove that | f | is also Riemann integrable. Also show that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

15

(b) Test for uniform convergence of the series $\sum_{n=0}^{\infty} x\,e^{-nx}$ in the closed interval [0, 1]. 15

(c) The only singularities of single valued function f(z) are poles of order 1 and 2 at $z=-1$ and $z=2$, respectively with residues at these poles 1 and 2 respectively. If $f(0)=\frac{7}{4}$ and $f(1)=\frac{5}{2}$, then determine the function and expand it by Laurent series in the annulus $1<|z|<2$. 15

(d) By using residues, show that

$$\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}dx=\pi.$$

15

7. (a) Let **f** be a continuous mapping of the closed interval [a, b] into $\mathbf{R}^n$ and **f** be differentiable in the open interval (a, b). Prove that there exists $x_0\in(a, b)$ such that :

$$|\mathbf{f}(b)-\mathbf{f}(a)|\leq(b-a)\,|\mathbf{f}'(x_0)|.$$

15

(b) Show that at any point on the equiangular spiral $r = a\, e^{\theta \cot \alpha}$, the radius of curvature $\rho = r \operatorname{cosec} \alpha$ and that it subtends a right angle at the pole. 15

(c) A circular arc revolves about its chord. Find the area of surface formed, if the radius of the arc is '**a**' and it subtends an angle 2α at the centre. 15

(d) Prove that $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$, $0 < n < 1$ and evaluate the integral

$$\int_0^\infty \frac{dx}{a^2 + x^4}.$$ 15

8. (a) Prove that

$$\vec{b} \cdot \nabla\left(\vec{a} \cdot \nabla \frac{1}{r}\right) = \frac{3(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r})}{r^5} - \frac{\vec{a} \cdot \vec{b}}{r^3},$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = r$

and $\vec{a}$ and $\vec{b}$ are constant vectors. 15

(b) Evaluate

$$\iint_S \vec{F} \cdot \hat{n}\, dS,$$

where $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and S is the surface of the solid bounded by the co-ordinate planes and the plane $x + y + z = a$ in the first octant. 15

(c) State and prove Serret-Frenet formulas for curves in space. 15

(d) Let $\vec{a}$ and $\vec{b}$ be two non-parallel unit vectors. If $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then prove that $|\vec{v}| = |\vec{u}| + |\vec{u} \cdot \vec{b}|$. 15