Mathematics

Paper-I

Time Allowed: Three Hours **Maximum Marks: 300**

Note: Candidate should answer questions No. 1 and 5 which are compulsory and any ***three*** *of the remaining questions, selecting at least* ***one*** *from each section.*

SECTION – A

1. Answer any **five** of the following: 12×5=60

(a) Find the eigen values and corresponding eigen vector of the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

(b) Show that if A is Hermitian and P is unitary, then $p^{-1}AP$ is Hermitian.

(c) Prove that $\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}(\sin\theta x + \cos\theta x)$.

(d) Evaluate $\underset{x\to 0}{Lim}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x}}$.

(e) Examine $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ for extreme values.

(f) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$ by changing to polar coordinates.

(g) Find the equation of the plane passing through the line $\frac{x-1}{4}=\frac{y-2}{6}=\frac{z-1}{3}$ and the point (4, 3, 7).

2. Answer all the **five** parts: 12×5=60

(a) Show that the set $\{x^3-x+1,\ x^3+2x+1,\ x+1\}$ is linearly independent set of vectors in the vector space of all polynomials over the field of real numbers.

(b) Prove that the set $\{(2,1,4),\ (1,-1,2),\ (3,1,-2)\}$ forms a basis of R^3.

(c) Reduce the matrix $A=\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ to column echelon form and hence determine its column rank.

(d) Verify Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ and compute A^{-1} also.

(e) Determine the definiteness of the following quadratic form in R^3:

$$6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_3x_1.$$

3. Answer all the **five** parts: 12×5=60

(a) Discuss the continuity and differentiability of the function $f(x)=|x-1|+|x-2|$ in the interval [0, 3].

(b) Find all the asymptotes of the curve

$$y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x+1=0.$$

(c) If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

(d) Find the values of x, y, z for which $\frac{5xyz}{x+2y+4z}$ is maximum given that $xyz=8$.

(e) Prove that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma p}{q^p}$ where $p > 0, q > 0$.

4. Answer all the **five** parts: 12×5=60

(a) Prove that the locus of a variable line which intersects the three lines:

$y = mx,\ z = c;\ y = -mx,\ z = -c;\ y = z,\ mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$.

(b) What conic does the equation $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$ represent? Find its centre.

(c) Find the equation of the sphere which passes through the four points $(0, -1, -2)$, $(-2, 1, 2)$, $(2, 2, -1)$, $(0, 2, 3)$ and hence find its centre and radius.

(d) Find the equation of the right circular cylinder of radius 2 and axis as the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

(e) Find the point of contact at which the plane $lx + my + nz = p$ touches the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

SECTION – B

5. Answer any **five** of the following: 12×5=60

(a) Solve the differential equation $y = 3x + \log p$.

(b) Solve the differential equation

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

(c) Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ given that $y = \left(x + \frac{1}{x}\right)$ is one integral.

(d) A particle moves with SHM in a straight line. In the first second after starting from rest, it travels distance 'a' and in the next second, it travels distance 'b' in the same direction. Prove that the amplitude of the motion is $\frac{2a^2}{3a-b}$.

(e) A cylindrical vessel on a horizontal circular base of radius 'a' is filled with a liquid of density ω with a height h. If a sphere of radius c and density greater than ω is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface.

(f) Find λ, μ and υ so that the vector $\vec{f} = (2x + 3y + \lambda z)\hat{i} + (\mu x + 2y + 3z)\hat{j} + (2x + \upsilon y + 3z)\hat{k}$ is irrotational.

(g) Find the curvature and torsion at any point of the curve $x = t, y = t^2, z = t^3$.

6. Answer all the **five** parts: 12×5=60

(a) Solve the differential equation $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$.

(b) Solve the differential equation $\sin px \cos y = \cos px \sin y + p$ and obtain the singular solution.

(c) Solve the differential equation $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = x^2 e^x$.

(d) Solve the differential equation:

$$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \cdot$$

(e) Apply the method of variation of parameters to solve:

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax .$$

7. Answer all the **five** parts: $12 \times 5 = 60$

(a) A particle is projected with a velocity υ so that its range in a horizontal plane is twice the greatest height attained. Show that the range is $\frac{4\upsilon^2}{5g}$.

(b) Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to the point from rest at a distance from the sun equal to the length of the major axis.

(c) One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. If ABC be a right-angle when the rod is on the point of sliding, μ the coefficient of friction and α the angle between AB and the vertical, show that $\mu = \frac{\tan\alpha}{2+\tan^2\alpha}$.

(d) Prove that the necessary and sufficient condition that a particle acted upon by a number of coplanar forces be in equilibrium is that the sum of the virtual works done by the forces in any small virtual displacement, consistent with the geometrical conditions of the system, is zero.

(e) An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse.

8. Answer all the **five** parts: $12 \times 5 = 60$

(a) Prove that the necessary and sufficient condition for the vector function $\vec{f}$ of a scalar variable t to have a constant magnitude is $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$.

(b) Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

(c) Verify Green's theorem in the plane for $\oint_c \left(xy + y^2\right)dx + x^2dy$ where c is the closed curve of the region bounded by $y = x$ and $y = x^2$.

(d) Show that the Serret-Frenet formulae can be written in the form

$$\frac{d\hat{t}}{ds} = \vec{\omega} \times \hat{t}, \frac{d\hat{n}}{ds} = \vec{\omega} \times \hat{n}, \frac{d\hat{b}}{ds} = \vec{\omega} \times \hat{b}.$$

Also determine the vector $\vec{\omega}$.

(e) Prove that the necessary and sufficient condition for the curve to be a plane curve is $\left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right] = 0$.