Mathematics

(Main Examination)

Paper-I

Time Allowed: Three hours **Maximum Marks: 300**

The figures in the margin indicate full marks for the questions.

Questions nos. **1** and **5** are compulsory. Candidates should answer any **three** questions from the rest, selecting at least **one** from each section.

SECTION-A

1. Answer any **five** of the following : 12×5=60

(a) Let V be a vector space over a field F. Suppose that

$S = \{x_1, x_2, \ldots x_n\}$

is a subset of non-zero elements of V. Then show that S is linearly dependent if and only if there is a k such that $1 \le k < n$ and

$x_{k+1} \in [\{x_1, x_2, \ldots, x_k\}]$

(b) Show that the matrix $\frac{1}{5}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$ is unitary, while the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is orthogonal.

(c) Examine the following function for maximum and minimum :

$y = 2\sin x + \cos 2x$

in the interval $[0, 2\pi]$.

(d) Compute the area of a region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

by using integration.

(e) Evaluate the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$$

(f) State and prove Taylor's theorem.

(g) Find equation of ellipse whose centre is at the origin and passes through the points (2, 2) and (3, 1).

2. Answer all the **five** parts : 12×5=60

(a) Prove that the four vectors $\alpha_1 = (1, 2, 3)$, $\alpha_2 = (1, 0, 0)$, $\alpha_3 = (0, 1, 0)$ and $\alpha_4 = (0, 0, 1)$ in $\mathbb{R}^3$ form a linearly dependent set.

(b) Find the rank of a matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

(c) Find the minimal polynomial of the following matrix :

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

(d) Find all the eigen values and corresponding eigen vectors of the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(e) State and prove Cayley-Hamilton theorem

3. Answer all the **five** parts : 12×5=60

(a) Find the derivative of the function

$$y = \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3\, e^x}$$

(b) State and prove mean-value theorem.

(c) Find the asymptotes of the curve

$$y = \frac{x^2 + 2x - 1}{x}$$

(d) Test the following function for maximum and minimum :

$$z = x^2 - xy + y^2 + 3x - 2y + 1$$

(e) Define Gamma and Beta functions and show that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

4. Answer all the **five** parts : 12×5=60

(a) Find the equation to the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

(b) Find equations of the line through the point (1, 1, 1) which meet both the lines

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-2}{4},\ x=2y=3z$$

(c) Find equations to the spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$ and touch the plane $4x+3y=15$.

(d) Find equation to the cone with vertex at the origin which pass through the curve given by

$x^2+y^2+z^2+2ax+b=0,\ lx+my+nz=p$

(e) Show that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$. Find the coordinates of the point of contact.

SECTION-B

5. Answer any **five** of the following : 12×5=60

(a) Solve the differential equation

$$\frac{dy}{dx}=-xy+x^3y^3$$

(b) Find complete solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$$

(c) Solve the differential equation

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

(d) Write the equation of motion of a particle moving along a straight line with known initial velocity and acceleration.

(e) One cubical box of one side equal to 'a' is placed on a fixed solid sphere such that the center of the lower face of the box is in touch with the highest point of the sphere. Find the minimum radius of the sphere for which the equilibrium is stable.

(f) Find the directional derivative of

$f(x, y, z)=xy^2+yz^3$ at $(2, -1, 1)$

in the direction of the vector $i+2j+2k$

(g) Show that

curl $(\phi \text{ grad } \phi)=\bar{0}$

6. Solve all the **five** parts : 12×5=60

(a) Solve the equation

$$\frac{dy}{dx} = \frac{2y}{x+1} + (x+1)^3$$

(b) Find the general and singular solutions of the equation

$$y = x\frac{dy}{dx} + a\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

(c) Solve the differential equation

$(1 + xy)\, x\, dy + (1 - xy)\, y\, dx = 0$

(d) Solve the differential equation

$$\frac{d^3y}{dx^3} + y = 3 + e^x + 5e^{2x}$$

(e) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

7. Solve all the **four** parts : 15×4=60

(a) In a Simple Harmonic Motion is amplitude is 'a' and period 'T' then find the relation between distance x from the center and velocity v at that point.

(b) Find the velocity of projection and direction of a bullet which just crosses a wall at 50 metres away and 25 m high in horizontal direction.

(c) If the maximum and minimum velocities of planet moving round the sun are respectively 30 and 29.2 km/sec, then find the eccentricity of the orbit.

(d) State and explain Bernoulli's equation.

8. Solve all the **five** parts : 12×5=60

(a) Prove the derivative of a constant vector is zero.

(b) Find the angle between the tangents to the curve $\bar{r} = t^2 i + 2tj - t^3 k$ at the points $t = 1$ and $t = -1$.

(c) Show that curl curl $\bar{f} = 0$ for $\bar{f} = zi + xj + yk$

(d) Evaluate $\iint_S \left[(x^3 - yz)\, dydz + z\, dxdy - 2x^2 ydzdx\right]$

over the surface bounded by the planes $x = y = z = a$.

(e) Verify Stoke's theorem for $\bar{f} = yi + zj + xk$ where S is the upper half of $x^2 + y^2 + z^2 = 1$ and C is its boundary.