# CC(M)

# MATHEMATICS

# (OPTIONAL)

# PAPER - I

# [37]

| *Time Allowed - Three Hours* | *Maximum Marks-250* |
|---|---|

## INSTRUCTIONS

***Please read each of the following Instructions carefully before attempting the paper.***

i) *There are **Eight** questions divided in **Two** Sections and printed in English. Candidate has to attempt **Five** questions in all. Question **No.1** and **5** are compulsory and out of the remaining, any **Three** are to be attempted choosing at least **One** question from each Section. The number of marks carried by a Question/Part is indicated against it. Answers must be written in English in Question-Cum-Answer(QCA) Booklet in the space provided.*

ii) *Your answer should be precise and coherent.*

iii) *If you encounter any typographical error, please read it as it appears in the text book.*

iv) *Candidates are in their own interest advised to go through the general instructions on the back side of the title page of the Answer Script for strict adherence.*

v) *No continuation sheets shall be provided to any candidate under any circumstances.*

vi) *No blank page should be left in between answers to various questions.*

## SECTION - A

1. a) Find the characteristic polynomial and minimal polynomial of the symmetric matrix $A$: $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$: Hence or otherwise prove that $A$ is similar to a diagonal matrix.

**(10)**

b) Find an integrating factor of $(y^2 + 2x^2y)dx - (xy - 2x^3)dy = 0$ and hence solve it. **(10)**

c) Find the volume enclosed by the surfaces $x^2 + y^2 = z$, $x^2+y^2 = 2x$, $z = 0$. **(10)**

d) If $f''(t) > 0, \forall t \in \mathbb{R}$, show that $f\left[\frac{1}{2}(x+y)\right] \leq \frac{1}{2}[f(x) + f(y)], \forall x, y \in \mathbb{R}$.

**(10)**

e) Show that the equation $x^2 + 2yz - 1 = 0$ represents a quadric of revolution. Also find the axis of revolution. **(10)**

2. a) Prove that the shortest distance from the origin to the hyperbola:

$x^2 + 8xy + 7y^2 = 225$, $z = 0$ is 5 units. **(20)**

b) Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = 2f'(x) + 3\int_0^x f(t)dt.$$

Verify the rank-nullity theorem for $T$, and prove that $T$ is one-to-one, but not onto. ($P_n(\mathbb{R})$ is the vector space of all polynomials with real coefficients having degree $\leq n$ ). **(15)**

c) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^{-2}e^{-x}$ by the method of variation of parameters. **(15)**

3. a) Discuss the convergence of the following improper integrals: **(20)**

i) $\int_0^{\pi/2} \frac{\sin^p x}{x^q} dx$

(ii) $\int_0^{\infty} \left(\frac{1}{1+x} - \frac{1}{e^x}\right) \frac{dx}{x}$.

b) Let $A$ be a unitary matrix such that $\lambda = -1$ is *not* an eigenvalue of $A$, then show that $A = (I + iH)^{-1}(I - iH)$, for some Hermitian matrix $H$. **(15)**

c) Find the equations of the line drawn parallel to $\frac{x}{4} = \frac{y}{1} = \frac{z}{1}$ so as to meet the lines **(15)**

$z = 5x - 6 = 4y + 3$

$z = 2x - 4 = 3y + 5.$

**4.** a) Investigate for what values of $\lambda, \mu$ the simultaneous non-homogeneous linear equations

$x + 2y + z = 8$

$2x + y + 3z = 13$

$3x + 4y + \lambda z = \mu$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions. **(20)**

b) Show that all the spheres that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ cut the coordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2 + y^2 + z^2 + 2bx - 2ay = 0$ **(15)**

c) Evaluate

$$\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{xy(x^2 + y^2)}{\sqrt{x^2 - (x^2 + y^2)^2}} \, dxdy.$$ **(15)**

## SECTION - B

**5.** a) Determine the orthogonal trajectories of the family of curves $y^2 - 2cx = c^2$. Is this family of curves self-orthogonal? Justify. **(10)**

b) If a particle moving along a space curve has velocity $\vec{V}$ and acceleration $\vec{f}$, show that the radius of curvature $\rho$ is given by $\frac{|\vec{v}|^3}{|\vec{v} \times \vec{f}|}$ **(10)**

c) Solve $\left(4y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2\right)\left(\dfrac{dy}{dx}\right)^2=3$ **(10)**

d) Show that the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid $ax^2+by^2+cz^2=1$ is a sphere centre at origin, the same as that of the given conicoid. What this sphere is called? **(10)**

e) A particle moves in a straight line towards a centre of force varying inversely as the cube of the distance from the centre, starting from rest at a distance $a$ from the centre of the force. Find its velocity and the time of reaching a point distant $b$ from the centre of force. **(10)**

**6.** a) Let $V$ and $W$ be finite - dimensional vector spaces (over the same field). Prove that $V$ is isomorphic to $W$ if and only if $\dim(V)=\dim(W)$. **(20)**

b) Find the locus of a line which meets the lines $y=mx$, $z=\pm c$ and the circle $x^2+y^2=1, z=0$. **(15)**

c) Use Laplace transforms to solve $\dfrac{d^2y}{dt^2}+4\dfrac{dy}{dt}+4y=t^3e^{-2t},\ y(0)=0,\ y'(0)=0.$ **(15)**

**7.** a) Show that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has its vertex at (2,4,1), in a rectangular hyperbola. **(20)**

b) Solve $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\dfrac{1}{(1-x)^2}$, $x>1$. **(15)**

c) A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are $\alpha$ and $\beta\,(\beta>\alpha)$. Show that the inclination of the beam to the horizontal in one of the equilibrium positions is $\tan^{-1}\left((\cot\alpha-\cot\beta)/2\right)$, and the beam is unstable in this position. **(15)**

**8.** a) A particle, inside and at the lowest point of a fixed smooth hollow sphere of radius $a$, is projected horizontally with velocity $\sqrt{7ga/2}$. Show that it will leave the sphere at a height $3a/2$ above the lowest point and that its subsequent path meets the sphere again at the point of projection. **(20)**

b) Solve $x(x\cos x-2\sin x)\dfrac{d^2y}{dx^2}+(x^2+2)\sin x\dfrac{dy}{dx}-2(x\sin x+\cos x)y=0.$ **(15)**

c) If $\vec{F}=(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}$, evaluate $\iint_S(\vec{\nabla}\times\vec{F}).\hat{n}\,dS$, where $S$ is the surface of the sphere $x^2+y^2+z^2=16$ above the $xy$ plane. **(15)**

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