Total No. of Printed Pages-4] **Roll No. ____________**

CC(M)

MATHEMATICS

(OPTIONAL)

PAPER - I

[37]

Time Allowed - Three Hours ***Maximum Marks-250***

INSTRUCTIONS

Please read each of the following Instructions carefully before attempting the paper.

i) *There are Eight questions divided in **Two** Sections and printed in English. Candidate has to attempt **Five** questions in all. Question **No.1** and **5** are compulsory and out of the remaining, any **Three** are to be attempted choosing at least **One** question from each Section. The number of marks carried by a Question/Part is indicated against it. Answers must be written in English in Question-Cum-Answer(QCA) Booklet in the space provided.*

ii) *Your answer should be precise and coherent.*

iii) *If you encounter any typographical error, please read it as it appears in the text book.*

iv) *Candidates are in their own interest advised to go through the general instructions on the back side of the title page of the Answer Script for strict adherence.*

v) *No continuation sheets shall be provided to any candidate under any circumstances.*

vi) *No blank page should be left in between answers to various questions.*

SECTION - A

1. a) Solve $\left(1+x\left(1-x^2\right)^{-\frac{1}{2}}-y\right)dx-\left(1-x^2\right)^{\frac{3}{2}}dy=0$ **(10)**

b) Find a basis and the dimension for the subspace of $\mathbb{R}^3$ given by $W=\left\{(x,\ y,\ z)\in\mathbb{R}^3:2x-3y+4z=0\right\}$ **(10)**

c) Evaluate $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$, where V is the region bounded by $z=x^2+y^2$ and $z=8-\left(x^2+y^2\right)$ **(10)**

d) A function $f(x)$ is twice differentiable and satifies the inequalities $|f(x)|<a^2, |f''(x)|<b^2,\ \forall x>a$ where a and b are constants. Prove that $|f'(x)|<2ab,\ \forall x>a$ **(10)**

e) Find the equation of the right circular cone with vertex (1, -2, -1), semi-vertical angle 60° and the line $\dfrac{x-1}{3}=\dfrac{y+2}{-4}=\dfrac{z+1}{5}$ as its axis. **(10)**

2. a) Find kernel, nullity, range and rank of the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ given by $T(x,\ y,z)=(x-y+2z,\ 2x+y,\ -x-2y+2z)$
Also verify the rank-nullity theorem for T. **(20)**

b) If $f(0)=0$, $f'(x)=\dfrac{1}{1+x^2}$. Using Jacobian method prove that $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$ **(15)**

c) Solve $y''+2y'+5y=17\cos 2x$, given that $y=3$ and $y'=0$ when $x=0$. **(15)**

3. a) Find the volume generated by the revolution of the loop of the curve $r\cos\theta=a\cos 2\theta$ about the initial line. **(20)**

b) Find the dimension and a basis of the solution space W of the system

$$\begin{aligned} x+2y+z-3w&=0\\ 2x+4y+4z-w&=0\\ 3x+6y+7z+w&=0 \end{aligned}$$ **(15)**

c) Find the equations of the line of the greatest slope on the plane $3x-4y+5z-5=0$ drawn through the point $(3,-4,-4)$; given that the plane $4x-5y+6z-6=0$ is horizontal. **(15)**

4. a) Find an orthogonal matrix which diagonalizes.

$$A=\begin{pmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{pmatrix}$$ **(20)**

b) A line of constant length has its extremities lie on the $y = x\tan\alpha,\ z = c;\ y = -x\tan\alpha,\ z = -c$. Prove that the locus of its middle point is an ellipse. **(15)**

c) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{dxdy}{\left(1+e^x\right)\sqrt{a^2-x^2-y^2}}$. **(15)**

SECTION - B

5. a) If u and v are two linearly independent integrals of $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ where P and Q are continuous functions of x. Prove that

$$u\frac{dv}{dx} - v\frac{du}{dx} = ce^{-\int Pdx}$$ (c is a non-zero constant) **(10)**

b) Show that if the differentiable scalar function $\phi(x, y, z)$ is any solution of Laplace's equation, then prove that grad ϕ is a vector that is both solenoidal and irrotational. Also prove that $\operatorname{div}(\phi\operatorname{grad}\phi) = |\operatorname{grad}\phi|^2$, and curl $(\phi\operatorname{grad}\phi) = \vec{0}$ **(10)**

c) Reduce the equation $axyp^2 + \left(x^2 - ay^2 - b\right)p - xy = 0$, where $p = dy/dx$ to Clairaut's form. Hence solve the equation and find its singular solution. **(10)**

d) Show that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z = 2$ in a circle of radius unity and also find the equation of the sphere which has this circle as one of its great circles. **(10)**

e) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point $P(OP = b)$ with velocity v in the direction OP. Prove that the time which elapses before it returns to P is $(T/\pi)\tan^{-1}(vT/2\pi b)$ **(10)**

6. a) Find the characteristic polynomial, eigenvalues and eigenvectors of the matrix A of the linear transformation $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$ with respect to the standard ordered basis for $P_2(\mathbb{R})$, vector space of all polynomials having degree less than or equal to 2 with real coefficients. **(20)**

b) Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1,\ x=0$; and parallel to the line $\frac{x}{a}-\frac{z}{c}=1,\ y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$, and if $2d$ is the shortest distance, Prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$. **(15)**

c) Use the Laplace transform to solve the initial-value problem.

$y''-3y'+2y=e^{-4t},\ y(0)=1,\ y'(0)=5$ **(15)**

7. a) Prove that the enveloping cylinders of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose generators are parallel to the lines

$x=0,\ \frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ meet the plane z=0 in circles. **(20)**

b) Solve $x^2\frac{d^2y}{dx^2}-(2m-1)x\frac{dy}{dx}+\left(m^2+n^2\right)y=n^2x^m\log x,\ x>0$ **(15)**

c) A smooth parabolic wire $(\text{latus rectum}=4a)$ is fixed with its axis vertical and vertex downwards and in it is placed a uniform rod of length $2l$ with its ends resting on the wire. Show that, for equilibrium, the rod is either horizontal, or makes with horizontal an angle $\cos^{-1}\sqrt{2a/l}$ **(15)**

8. a) A particle of unit mass moves under a central force $\mu r\left(r^4-c^4\right)$, and is being projected from an apse at a distance c with velocity $c^3\sqrt{2\mu/3}$. Show that its path is $x^4+y^4=c^4$ **(20)**

b) Solve by the method of variation of parameters.

$(1-x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=2(x-1)^2e^{-x},\ 0<x<1$ **(15)**

c) Let ϕ and ψ be two scalar valued functions of position with continuous derivatives of second order at least. Prove that $\iiint_V\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)dV=\iint_S(\phi\nabla\psi-\psi\nabla\phi).\hat{n}dS$ where V is the volume bounded by a closed surface S and $\hat{n}$ is the positive normal to S. **(15)**