

1(CCE-M)6
MATHEMATICS - II

[15]

Time Allowed -3 Hours

Maximum Marks-300

INSTRUCTIONS

- i) *Answers must be written in English*
- ii) *The number of marks carried by each question is indicated at the end of the question.*
- iii) *The answer to each question or part thereof should begin on a fresh page.*
- iv) *Your answer should be precise and coherent.*
- v) *The part/parts of the same question must be answered together and should not be interposed between answers to other questions.*
- vi) *Candidates should attempt any **five** questions*
- vii) *If you encounter any typographical error, please read it as it appears in the text book.*
- viii) *Candidates are in their own interest advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.*
- ix) *No Continuation sheets shall be provided to any candidate under any circumstances.*
- x) *Candidates shall put a cross (X) on blank pages of answer Script.*
- xi) *No blank page be left in between answer to various questions.*
- xii) *No programmable Calculator is allowed.*
- xiii) *No stencil (with different markings) is allowed.*
- xiv) *In no circumstances help of scribe will be allowed.*

Section - A

- 1. a) Suppose G is a group and $g \neq e$ is an element of G . Under what conditions on g is there a homomorphism $f : \mathbb{Z}_{15} \rightarrow G$ with $f([1]) = g$? Under what conditions on G is there an injective homomorphism $f : \mathbb{Z}_{15} \rightarrow G$? (20)
- b) Show that every group of order 85 is cyclic (20)
- c) Using Sylow Theorems, show that in a group of order pq with two primes p, q and $p < q$, there is only one subgroup of order q . Further show that this subgroup is normal. (20)

2. Let G be an abelian group of order n .

a) If $f: G \rightarrow \mathbb{C}$ is a function, then prove that for all $h \in G$,

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(hg) \quad (20)$$

b) Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers and suppose

$f: G \rightarrow \mathbb{C}^*$ is a homomorphism. Prove that

$$\sum_{g \in G} f(g) = 0 \text{ or } \sum_{g \in G} f(g) = n \quad (30)$$

c) If $f: G \rightarrow \mathbb{C}^*$ is any homomorphism, then prove that $\sum_{g \in G} |f(g)| = n$ (10)

3. Let $R_n = \mathbb{Z}_n[x]$ be the polynomial ring over integers modulo n .

a) In the field $R_2 / (x^2 + x + 1)$ let α be the image of x and compute (in reduced

$$\text{form}) (1 + \alpha - \alpha^2)^{-1} \quad (30)$$

b) In the field $R_3 / (x^3 - x + 1)$, let α be the image of x and compute (in reduced

$$\text{form}) (1 + \alpha)(2 + \alpha - \alpha^2) \quad (30)$$

4. Let (X, d) be a metric space

a) Prove that $d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ is metric on X . (10)

b) Let A be a nonempty subset of X . For any $x \in A$, we define $d(x, A) = \inf \{d(x, y) : y \in A\}$.

Prove that for any $x, y \in X$, we have $|d(x, A) - d(y, A)| \leq d(x, y)$ (30)

c) Let $a, b \in \mathbb{R}, a < b$ and let

$$C([a, b]) = \{f: [a, b] \rightarrow \mathbb{C} : f \text{ is continuous on } [a, b]\}$$

Show that the map $d: C([a, b]) \times C([a, b]) \rightarrow \mathbb{R}_+$ defined by

$$d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \quad \text{is a metric on}$$

$$C([a, b]). \text{ Therefore, } (C([a, b]), d) \text{ is a metric space} \quad (20)$$

5. a) Prove that if f is holomorphic in the annulus

$$U = \{z \in \mathbb{C} : r < |z - a| < R\},$$

then the integral $I = \int_{|z-a|<\rho} f(z) dz$

has the same value for all ρ with $r < \rho < R$.

(30)

- b) Compute the value of the integral

$$I = \int_{\gamma} \frac{z+1}{z(z-1)} dz$$

Where γ is the circle $|z-2| = \sqrt{2}$, traversed counterclockwise.

(30)

6. a) Find the radius of convergence for the series

$$\sum_{n \geq 1} \frac{n^n z^n}{n!} \text{ and } \sum_{n \geq 1} 2^{-n^2} z^n$$

(30)

- b) For any positive integer n , prove that

$$\int_0^{2\pi} (1 + 2 \cos t)^n (\cos nt) dt = 2\pi$$

(30)

7. a) Let $f_n(x) = x^n$ for $x \in [0, 1]$. Then $\{f_n\}$ converge pointwise but not uniformly.

on $[0, 1]$. Let $g : [0, 1] \rightarrow \mathbb{C}$ be a continuous function such that $g(1) = 0$. Prove that $\{g(x)x^n\}$ converge uniformly on $[0, 1]$

(30)

- b) For $f(t) = 2t^2 + 1$, $\alpha(t) = t + [3t]$ on $[0, 1]$ and P is the partition of $[0, 1]$ consisting of 4 subintervals of equal length, find $U(P, f, \alpha)$ and $L(P, f, \alpha)$. Further.

calculate $\int_0^1 \int d\alpha$ if it exists.

(30)

8. a) Use separation of variables to solve the wave equation $u_{xx} - C^2 u_{tt} = 0$ with initial conditions $u(x, 0) = f(x)$, $u_x(x, 0) = g(x)$.

(20)

- b) Use the method of characteristics to solve the wave equation and verify that these solutions agree with those obtained in part (a). (20)
- c) Find the general integral of the PDE :

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = 3. \quad (20)$$

SECTION - B

1. a) Derive the Euler-Lagrange equations for the motion of a rigid body under a conservative force. (20)
- b) Consider a simple pendulum of length l vertically suspended from a point. If a horizontal force of $F \cos(\omega t)$ is applied at that point, prove that the pendulum settles into a periodic motion. Find the period of this motion. (20)
- c) Consider a circular turn table inclined at angle of θ to the horizontal, which has a constant angular velocity ω about the fixed centre. A block of wood of mass m lies at rest relative to the turn table. What is the smallest value of θ for which the block will begin to slide at atleast one point in the period of rotation. Assume that the coefficient of friction between the table and the block is μ . (20)
2. a) Describe the test for incompressibility of a flow. How are the stream lines derived from a given flow. (20)
- b) Derive Navier Stokes equation for an incompressible flow. (20)
- c) Give the definitions of the terms source and sink. How do the flow equations change in the presence of these phenomena. (20)
3. a) Find a real root of the polynomial $p(x) = x^5 + 3x^3 - 1$, correct to $\epsilon = 0.05$ using the regula falsi method as well as the Newton-Raphson method. Which has higher convergence and why? (30)
- b) Derive the cubic spline interpolation for the function given by the following table

x	1	3	5	7
$f(x)$	2	6	24	120

Use this to find the values of $f(2)$ and $f(6)$. (30)

4. a) Solve the I.V.P. $y=(1+y)^2$, $y(0)=1$, in the interval $0 \leq x \leq 1$ by using the Runge Kutta method of order 2 by using $h = 0.2$ (30)
- b) State the n-point Gaussian quadrature formula for weight $w(x)=1$ in the interval $[-1,1]$. Derive the error of approximation for any continuous function $f \in C^{n+1}([-1,1])$. (30)
5. a) A token collector collects tokens number $1,2,\dots,n$ all of which are equally likely. What is the expected number of tokens collected before token number i is collected. What are the expected number of tokens before the collection is complete (viz., at least one token of every number is collected). (30)
- b) Consider a random walk in the set of integers, where the starting point, is 0 and the steps taken at any point are independently chosen from either of 0, $\pm 1, \pm 2$ with equal probability. Let X be the random variable that holds the value of the current position after n steps. Find $E[X]$ and V ar $[X]$. Hence, use the central limit theorem to find the limiting distribution of X as $n \rightarrow \infty$. (30)
6. a) Which continuous distribution has the memory-less property? Give a proof of this statement. (10)
- b) What is the difference between Multiple correlation coefficient and Rank correlation coefficient? Give an example where these values might be different. (20)
- c) Describe the least squares approach to linear regression. Derive the optimal parameters from first principles. (30)

