 Roll No. ______________

1(CCE-M)6

MATHEMATICS-I

[15]

Time Allowed -3 Hours *Maximum Marks-300*

INSTRUCTIONS

i) *Answers must be written in English.*
ii) *The number of marks carried by each question is indicated at the end of the question.*
iii) *The answer to each question or part there of should begin on a fresh page.*
iv) *Your answer should be precise and coherent.*
v) *The part/parts of the same question must be answered together and should not be interposed between answers to other questions.*
vi) *Candidates should attempt any* ***Five*** *questions.*
vii) *If you encounter any typographical error, please read it as it appears in the text book.*
viii) *Candidates are in their own interest are advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.*
ix) *No Continuation sheets shall be provided to any candidate under any circumstances.*
x) *Candidates shall put a cross (X) on blank pages of Answer Script.*
xi) *No blank page be left in between answer to various questions.*
xii) *No programmable Calculator is allowed.*
xiii) *No stencil (with different markings) is allowed.*
xiv) *In no circumstances help of scribe will be allowed.*

MATHEMATICS-I (12 Questions)

All questions carry same marks (60 each) and each part of every question carry the same weightage

1. Let U and W be subspaces of a vector space V. We define the sum U+W as follows : $U+W=\{u+w : u \in U, w \in W\}$.

   a) Show that U+W is a subspace of V containing both U and W.

   b) Show that span $\{u_1, u_2, ..u_r, w_1, w_2,...w_s\}$= span $\{u_1, u_2,..., u_r\}$ + span $\{w_1, w_2,..., w_s\}$for any vectors u_i's and w_j's t from U and W respectively.

c) If U and W are finite dimensional and if $U \cap W = \{0\}$, then show that dim(U+W) = dim (U) + dim (W).

2. Let $T:V \to V$ be a linear transformation of a finite dimensional vector space V.

a) Prove that if $U \subset V$ is a vector subspace, then $T(U) = \{T(u) : u \in U\}$ is a vector subspace of V.

b) Prove that if T is onto, it sends linearly independent set of vectors to linearly independent set of vectors.

c) Prove that if T is invertible and $U \subseteq V$ then dim T(U) = dim U.

3. Let A and B be m×n matrices.

a) Show that rank (A+B) $\leq$ rank (A) + rank (B)

b) Show that null (A) = null (UA) for any invertible m×m matrix U.

c) Show that dim (null (A) = dim (null (AV)) for any invertible n×n matrix V.

4. A complex matrix S is called skew-Hermitian,if $S^* = -S$

a) Show that $Z - Z^*$ is skew-Hermitian for any square complex matrix Z.

b) If S is skew-Hermitian, then show that S^2 and iS are Hermitian.

c) If S is skew-Hermitian, then show that the eigenvalues of S are purely imaginary ($i\lambda$ *for real* λ).

5. Find the equation (s) of the tangent (s).

a) to the parabola $y^2 = 2ax$ at the vertex and at the ends of Latus rectum.

b) to the hyperbola $x^2 - 2y^2 = 1$ from the point (7,5).

c) to the ellipse $\frac{x^2}{32} + \frac{y^2}{18} = 1$ at the point whose x coordinate is 2.

6. Prove the following :

a) Let $A \subset \mathbb{R}$. We say that a function $\int : A \to \mathbb{R}$ is Lipschilz on A if there exists L > 0 such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. Show that any Lipschitz function is continuous.

such that $|f(x)|>\dfrac{|f(c)|}{2}$ for all $x\in(c-\delta,c+\delta)\cap A$.

c) Let $f:[0,2\pi]\to[0,2\pi]$ be continuous such that $f(0)=f(2\pi)$. Show that there exists $x\in[0,2\pi]$ such that $f(x)=f(x+\pi)$.

7. a) Use Lagrange multipliers to find the maxima and minima of the function

$f(x,y)=x^2+2y^2$ subject to the constraint $x^2+y^2=1$

b) Find all the stationery points and local maxima and minima of the function

$f(x,y)=e^{x+y}(x^2+y^2-xy)$.

8. Let $F(y)=\int_0^{\infty}\dfrac{\sin xy}{x(x^2+1)}dx \;\; if \; y>0$

a) show that F satisfies the differential equation $F''(y)-F(y)+\pi/2=0$

b) Deduce that $F(y)=\dfrac{1}{2}\pi(1-e^{-y})$

c) Deduce that for $y>0 \; and \; a>0$ $\int_0^{\infty}\dfrac{\cos xy}{x^2+a^2}dx=\dfrac{\pi e^{-ay}}{2a}$

9. a) Solve the differential equation $y\cos x+2xe^y+(\sin x+x^2e^y-1)y'=0$

b) A radioactive substance obeys the equation $y'=ky$ where k<0 and y is the mass of the substance at time t. Suppose that initially, the mass of the substance is y(0) =M>0. At what time does half of the mass remain?

c) The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14; how much is left after 6000 years? how long do we have to wait before there is less than 2 milligrams?

10. Consider the initial value problem $y'+\dfrac{y}{4}=3+2\cos 2t, y(0)=y_0$

a) Find the solution to the homogenous equation of this problem.

b) Find the solution of this initial value problem and describe its behaviour for large t.