60-62nd C.C. Exam., 2018

MATHEMATICS

Time : 3 Hours] **[Max. Marks : 300**

*Candidates are required to answer six questions in all, selecting **three** from each Section.*

SECTION – I

1. (a) If $U(F)$ and $V(F)$ are two vector spaces and T is a linear transformation from U into V, then show that the range of T is a subspace of V.

(b) Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(x, yz) = (x + 2y - z,\ y + z,\ x + y - 2z)$$

Find the basis and dimension of (i) the range of T and (ii) the null space of T.

2. (a) Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{x}}$.

(b) Show that the function defined by $f(x) = [x - 2] + [x - 3]$ is continuous at 2 and 3.

3. (a) If $u = \tan^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$ then use Euler's theorem to prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$$

(b) Evaluate over S, $\iint xdydz + dzdx + xz^2dxdy$ where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

4. (a) Find the asymptotes of the hyperbola

$$x^2 + 3xy + 2y^2 + 2x + 3y = 0.$$

(b) Find the equation of hyperbola whose asymptotes are $x + 2y + 3 = 0, 3x + 4y + 5 = 0$ and passses through the point $(1, -1)$.

5. (a) Show that the function f, *where*

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0 \quad , & \text{if } x = y = 0 \end{cases}$$

is continuous, possesses partial derivatives but is not differentiable at the origin.

(b) Examine for conergence the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$$

6. (a) Prove that the thrust of a heavy homogeneous liquid on a plane area is equal to the product of the area and the pressure at its centroid (CG).

(b) Find the centre of pressure of a triangular area immersed in a liquid with its vertex in the surface and base horizontal.

SECTION – II

7. (a) Prove that every permutation in S_n, $n > 1$, is a product of two cycles.

(b) Let $|G| = 2p$, where p is an odd prime. Then prove that G is isomorphic to Z_{2p} or D_p.

8. (a) Show that the sequence f_n, where $f_n(x) = nx\,e^{-nx^2}$, $x \geq 0$ is not uniformly convergent on $[0, k]$, $k > 0$.

(b) Prove that continuous image of a compact set is compact.

9. (a) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$ where C is the circle $|z| = 3$.

(b) Evaluate the singularities of the given function

$$f(z) = \frac{z \cos\left(\frac{\pi z}{2a}\right)}{(z-a)(z^2+b^2)^7 \sin^5 z}$$

where a and b are distinct non-zero real numbers.

10. (a) Solve : $p^2 + q^2 = 1$

(b) Find the moment of inertia of a spherical shell

11 (a) Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ on the interval [0, 0·4]. Use the second-order implicit Runga-Kutta method.

(b) The marks obtained by a number of students for a certain subject are assumed to be approximately normally distributed with mean value 65 and with a standard deviation of 5. If 3 students are taken at random from this set, what is the probability that exactly 2 of them will have marks over 70 ?

12. (a) If variance of X, Y, X – Y are respectively, σ^2_X, σ^2_Y, σ^2_{X-Y}, then find coefficient of correlation r between two variables X and Y.

(b) By graphical method, find the maximum value of $Z = 2x + 3y$ subject to the constraints $x + y \leq 30$, $y \geq 3$, $0 \leq y \leq 12$, $x - y \geq 0$, $0 \leq x \leq 20$.