47th C.C. Exam., 2007

MATHEMATICS, PAPER – I

Time : 3 Hours] [Max. Marks : 200

Answer any five questions.

1. (a) Show that the mapping $T: V_2(F) \to V_3(F)$ defined by :
$T(a, b) = (a + b, a - b, b)$
is a linear transformation. Find range, rank, null space and nullity of T and verify rank-nullity theorem.

(b) When is a matrix called row-reduced echelon matrix ? Reduce the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ into row-reduced echelon matrix.

2. (a) Prove that continuity at any point is necessary but not a sufficient condition for differentiability at that point.

(b) If $u = \log_e(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{1}{(x + y + z)^2}.$$

(c) Trace the curve : $y^2(a+x) = x^2(a-x)$, $a > 0$ and find the area of the loop.

3. (a) Expand $2x^3 + 7x^2 + x - 6$ by Taylor's theorem in powers of $(x-2)$.

(b) Discuss the continuity of the function

$$f(x) = \frac{\frac{1}{e^x} - \frac{1}{-e^x}}{\frac{1}{e^x} - \frac{1}{+e^x}}, \text{ when } x \neq o; f(o) = 1.$$

(c) If $f(x) = (x-1)(x-2)(x-3)$ defined in $[o, 4]$ find the value of ξ such that $f(\xi)$ has the same value as the slope of the chord joinint point for which $x = 0$ and $x = 4$.

4. (a) If u, v, w are the roots of the cubic

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, then prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

(b) Prove that the maximum and minimum values of

$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, when $lx + my + nz = 0$ and

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are given by :

$$\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0$$

(c) Find the asymptotes of the curve :

$$x^2 y - xy^2 + xy + y^2 + x - y = 0$$

show that the asymptotes cut the curve again in three points, which lie on the straight line $x + y = 0$.

5. (a) Applying the definition of definite integral as a limit of a sum,

evaluate $\underset{n \to \infty}{Lt} \left(\frac{\underline{|n}}{n^n} \right)^{1/n}$.

(b) Change the order of integration in : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$ and evaluate it.

(c) Define Beta and Gamma functions. Prove that

(i) $\int_0^1 x^m (\log x)^n dx = \frac{(-)^n n}{(m+1)^{n+1}}$

(ii) $B(m-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$.

6. (a) Solve : $(xy \sin xy + \cos xy)\, y\, dx + (xy \sin xy - \cos xy)\, x\, dy = 0$.

(b) Find the orthogonal trajectories of confocal and coaxial parabolas $r = 2a\,(1 + \cos\theta)^{-1}$, where a is the parameter.

(c) Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$.

7. (a) Show that the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1, y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$.

(b) Prove that the locus of the middle points of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subtending a right angle at the centre, is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2.$$

(c) A sphere of constant radious k passes through origin. If the sphere cuts the axes in A, B, C, show that the locus of centroid of trangle ABC, is $9(x^2 + y^2 + z^2) = 4k^2$.

8. (a) State and prove Frenet-Serret formulae.

(b) Define curvature and torsion of a curve. Find the curvature of the helix : $x = a \cos\theta, y = a \sin\theta, z = o\,\theta \tan\alpha$ hence obtain torsion as ±.

9. (a) Define christoffel symbols of the first and second kinds. Show that they are not tensors.

(b) Show that the covariant derivative of

(i) the metric tensor

(ii) kronecker delta are identically zero.

(c) (i) If $\bar{r} = a\sin t\,\hat{L} + a\cos t\,\hat{j} + (at\tan x)\,\hat{K}$, find the value of

$$\left|\frac{d\bar{r}}{dt} \times \frac{\dot{d}\bar{r}^2}{dt^2}\right|.$$

(ii) Prove that : $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$.

10. (a) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform ords each of length b and weight w, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is

$$\frac{2w(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}.$$

(b) A uniform chain of length L and weight w hangs between two fixed points at the same level and weight w' is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of the support, is $\frac{k}{2l}w + \frac{l}{4k}w' + \frac{l}{8k}w$.

11. (a) A particle whose mass is m, is acted upon by a force $m\mu\left(x + \frac{a^4}{x_3}\right)$ towards the origin; if it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\Pi}{4\sqrt{\mu}}$.

(b) A particle of mass m moves under a central force $m\left(\frac{9}{r^4} - \frac{10}{r^5}\right)$ and is projected from an apse at a distance 5 units with speed