(b) Suppose U and W are subspaces of the vector space $IR^4(IR)$, generated by the sets
$B_1 = (1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)$
$B_2 = (1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)$ respectively.
Determine : (i) $dim\ (U + W)$ (ii) $dim\ (U \cap W)$.

2. (a) Find an orthogonal matrix A whose first row is $u_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

(b) Let f be the bilinear form on IR^2 defined by
$f\{(x_1, x_2), (y_1, y_2)\} = 2x_1y_1 - 3x_1y_2 + x_2y_2$

(i) Find the matrix A of f in the basis $\{u_1 = (1, 0), u_2 = (1, 1)\}$.

(ii) Find the matrix B of f in the basis $\{v_1 = (2, 1), v_2 = (1, -1)\}$.

(iii) Find the matrix P from the basis $\{u_i\}$ to the basis $\{v_i\}$ and verify that $B = P^t AP$.

3. (a) Show that $\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$; if $0 < u < v$ and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

(b) Find all the asymptotes of the curve
$x^2y^2(x^2 - y^2)^2 = (x^2 + y^2)^3$.

(c) If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

4. (a) Find the volume of the solid generated by the revolution of the curve $(a - x)\, y^2 = a^2x$ about its asymptote.

(b) Evaluate $\iint_C xy(x + y)\, dx\, dy$, where C is the area between two curves $y = x^2$ and $y = x$.

(c) Prove that : $B(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m + n)}$.

5. (a) A is a point on OX and B on OY so that the angle OAB is constant $(=\alpha)$. On AB as diameter a circle is described whose plane is parallel to OZ. Prove that as AB varies, the circle generates the cone $2xy - z^2 \sin 2\alpha = 0$.

(b) Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ intersects in a straight line lying in the plane $x = 0$. Show that the straight line touches the parabola.

6. (a) Solve : $\sin y \frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$.

(b) Solve : $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 19\frac{dy}{dx} + 20y = xe^x + 2e^{-4x}\sin x$.

(c) Find the differential equation of all tangent lines to the parabola $y = x^2$.

7. (a) Find the curvature and torsion of the curve
$x = a\cos\theta,\ y = a\sin\theta,\ z = b\theta$
where a and b are two constans.

(b) If $u_n = \int_0^{\frac{\pi}{2}} x^n \sin x\, dx$ and $n > 1$ then prove that

$$u_n + n(n-1)u_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$$

(c) Discuss the continuity of f at $x = 0$, when

$$f(x) = \begin{cases} x \log \sin x & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$$

8. (a) If a and b are constant vectors, then prove that
$grad\,[(r \times a) . (r \times b)] = (b \times r) \times a + (a \times r) \times b$.

(b) Evaluate $\int_C F.dr$, where C is the line joining $(0, 0, 0)$ and $(2, 1, 1)$ and $F = (2y + 3)\,i + xzj + (yz - x)\,k$.

(c) If $a = \sin\theta\, i + \cos\theta\, j + \theta k$
$b = \cos\theta\, i + \sin\theta\, j + 3k$
$c = 2i + 3j - k.$

then find $\frac{d}{d\theta}\{a \times (b \times c)\}$ at $\theta = 0$.

9. (a) Define the covariant derivative of a covariant vector field with respect to the connection coefficients L^i_{jk} and show that they are the components of a tensor of type (0, 2).

(b) (i) Show by an example that the two operations of contraction and covariant differentiation are commutative.

(ii) Define an inner product ϕ and the metric tensor g associated with ϕ.

(c) Define a tensor of tyep (2, 0) and give an example.

10. (a) A uniform chain of length $2l$ is to be suspended from two points A and B in the same horizontal line, so that either terminal tension is n times of that at the lowest point. Show that the span AB is $\frac{2l}{\sqrt{n^2-1}} \log\left(n + \sqrt{n^2-1}\right)$.

(b) A light ladder is supported on a rough floor and leans against a smooth wall. How far up the ladder can a man climb without slipping taking place ?

11. (a) A particle moves in a straight line with S.H.M. Its speeds at distances x_1, x_2 from the centre of the path are v_1, v_2 respectively.

Show that the period of the motion is $2\pi\left\{\frac{(x_2^2 - x_1^2)}{(v_1^2 - v_2^2)}\right\}^{1/2}$ and

its amplitude is $\left\{\frac{(v_1^2 x_2^2 - v_2^2 x_1^2)}{(v_1^2 - v_2^2)}\right\}^{1/2}$.

(b) Particles A and B are projectd in the same vertical plane from two points $(a, 0)$ and $(0, b)$ at angles α and β to the horizontal.