45th C.C. Exam., 2002

MATHEMATICS, PAPER – I

Time : 3 Hours] [Max. Marks : 200

Answer any five questions.

1. (a) If W is a subspace of a finite dimensional vector space $V(F)$, then prove that :

$$\dim \frac{V}{W} = \dim V - \dim W.$$

(b) Determine the eigen values and the corresponding eigen vectors of the following matrix : $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2. (a) Expand $\log_e \sin x$ in the powers of $(x-2)$.

(b) Find the envelope of the curves.

$\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$, when $a^p + b^p = c^p$, being any constant.

(c) Show that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

3. (a) Evaluate : $\lim\limits_{x \to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x}$.

(b) If a function $f(x)$ is as

(i) continuous in $[a, b]$

(ii) differentiable in (a, b).

then show that there exists at least one point $c \in (a, b)$ such that $f(b) = f(a) + (b-a)f'(c)$.

(c) Show that : $\dfrac{x}{1+x} < \log(1+x) < x$, for $x > 0$.

4. (a) Evaluate : $\displaystyle\int_0^{\pi} \frac{x\,dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$.

(b) Find the area common to the following two circles :

$r = a\sqrt{2}$ and $r = 2a\cos\theta$.

(c) Find the length of the arc of the parabola $y^2 = 4ax$ cut of by its latus rectum.

5. (a) Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.

(b) Prove that : $B(n, n) = \dfrac{\sqrt{\pi}\,\overline{|n}}{2^{2n-1}\sqrt{\left(n+\frac{1}{2}\right)}}$.

6. (a) Solve : $(1+y^2)dx = (\tan^{-1}y - x)\,dy$.

(b) Find the orthogonal trajectory of the family of curves $x^2 + y^2 + 2fy + 1 = 0$, where f is parameter.

(c) Solve : $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x}\sin x$.

7. (a) Prove that the equation :

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represents a pair of planes and find the angle between them.

(b) POP' is a variable diameter of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z = 0$ and a circle is described in the plane $PP'ZZ'$ and PP' as diameter. Prove that as PP' varies, the circle generates the surface

$$(x^2 + y^2 + z^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = x^2 + y^2.$$

8. (a) Show that the plane $8x - 6y - z = 5$ touches the paraboloid $\dfrac{x^2}{2} - \dfrac{y^2}{3} = z$ and find its point of contact.

(b) Prove that the tangent planes to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} + 1 = 0$ which cut $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1 = 0$ in ellipses of constant area πk^2 have their points of contact on the surface

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{k^4}{4a^2b^2c^2}.$$

9. (a) Prove that a skew symmetric tensor of rank two has $\dfrac{1}{2}N(N-1)$ independent component in V_N.

(b) Prove that the transformations of tensors form a group.

(c) Show that the covariant derivative of a mixed tensor of rank two is a tensor of rank three.

10. (a) If $\bar{a}, \bar{b}, \bar{c}$ be three unit vectors such that $\bar{a} \times (\bar{b} \times \bar{c}) = \dfrac{\bar{b}}{2}$, find the angles which $\bar{a}$ makes with $\bar{b}$ and $\bar{c}, \bar{b}$ and $\bar{c}$ being non-parallel.