44th C.C. Exam., 2002

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions.

1. (a) Define the rank of a matrix. Prove that the system of linear equatior $AX = B$ has a solution, if and only if, the column vector B is a linear combination of the columns of the matrix A i.e. the coefficient matrix of A and the augmented matrix (A, B) have the same rank.

(b) (i) State Cayley-Hamilton Theorem.

(ii) Find the polynomial for which the matrix

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix} \text{ is a root.}$$

(ii) Find all the eigen values and a basis for each subspace for

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

2. (a) Use Lagrange's Mean ValueTheorem to deduce that

$$x < \log \frac{1}{1-x} < \frac{x}{1-x}, \text{ for } 0 < x < 1.$$

(b) Find the extreme values of $a \cos x + b \cos 2x$, $a > 0$, $b > 0$,

where $-\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

(c) Show that the sequence $\{s_n\}$ defined by

$$s_{n+1} = \frac{6(1+s_n)}{7+s_n}, s_1 > 0 \text{ is monotonic and converges to 2.}$$

3. (a) Prove that : $\lim\limits_{x \to 0+} \frac{x-1+|x-1|}{x^2-1} = 1.$

(b) Draw the graph of the function $y = x\left(|x-1| + \frac{x}{3} - 1\right)$.

Determine the point, if any, at which this function is not differentiable.

(c) Find all the asymptotes to the graph of the function

$$f_1(x) = \frac{1}{x^2-1}$$

4. (a) Let $f(x) = \int_1^x \frac{(\log t)}{(t+1)} dt$ if $x > 0$. Find $f(x) + f\left(\frac{1}{x}\right)$.

(b) Show that $\int_0^{2\pi} \dfrac{x \sin 2x \sin\left(\dfrac{\pi}{2}\cos x\right)}{2x-\pi}\,dx = \dfrac{8}{\pi^2}$.

(c) Find the greatest and the least value of the function $f(x) = x^3 - 6x^2 + 9x + 1$, on the interval [0, 2].

5. (a) Prove that $(m+n+1)\,\beta\,(m+1, n+1) = n\beta\,(m+1, n)$ if

$\beta\,(m+1, n+1) = \int_0^1 x^m (1-x)^n\,dx$, m and n positive integers.

Deduce $\beta(m+1, n+1) = \dfrac{\lfloor m \lfloor n}{\lfloor m+n+1}$.

(b) A figure bounded by the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about the x-axis. Find the volume of the solid of revolution.

6. (a) Solve : $(xy^2 - 1)\,dx + y(x^2 - 1)\,dy = 0$.

(b) Solve : $x^4 k\dfrac{d^4y}{dx^4} + 2x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$.

(c) Find the orthogonal trajectories of the cardioid $r = a(1 - \cos\theta)$.

7. (a) Find the equation of the normal at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$.

(b) If the normals at two points A and B of a parabola intersect on the curve, show that the line AB passes through a fixed point.

8. (a) Find the length of the shortest distance between the lines

$\dfrac{x-3}{2} = \dfrac{y-5}{2} = \dfrac{z+1}{2}$ and $y - z = 1, x - 2 = 0$.

Find also the co-ordinates of the points where the line of shortest distance meets the given lines.

(b) Obtain the equation of the cone whose vertex is the point (α, β, γ) and base is the conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$

9. (a) Prove that every tensor of second rank can be represented in the form of a sum of a symmetric and skew-symmetric tensors which are uniquely determined by that tensor.

(b) Prove that $g_{ij}e^j = e_j, g^{ij}e_j = e^i$ where the symbols have usual meanings.

(c) Derive transformation locus for the Christoffel symbols of first kind.

10. (a) Show that $(\alpha \times \beta) \times (\gamma \times \delta) = [\alpha\, \gamma\, \delta]\, \beta - [\beta\, \gamma\, \delta]\, \alpha$ where $[\alpha\, \beta\, \gamma]$ denotes the products $\alpha \cdot [\beta \times \gamma]$.

(b) Find a three-dimensional vector field f such that div $f(x) = 2x_1 + x_2 - 1$ and curl $f(x) = u_3$. u_1, u_2, u_3 are the basis vectors.

(c) Prove that $u_1(x, y) = \dfrac{16a^2b^2}{\pi^4(a^2 + b^2)} \sin\dfrac{\pi x}{a} \sin\dfrac{\pi y}{b}$ satisfies the equation $-\nabla^2 u = 1, u(\pm a, y) = 0, u(x, \pm b) = 0$.

11. (a) A particle describes a plane curve under an acceleration f which is always directed towards a fixed point in the plane. Prove that in usual notations

(i) $vp = h$ (ii) $f = \dfrac{h^2}{p^3} \dfrac{dp}{dr}$.

(b) A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity, μ being a constant. Show that the distance it has described in time t is $\dfrac{1}{\mu V}\left[\sqrt{1+2\mu V^2 t} - 1\right]$.

12. (a) State and prove the principle of virtual work for a system of forces acting on a rigid plane lamina in its plane.

(b) A plane area is immersed in a homogeneous liquid with its centre of gravity and pressure at depths a and b respectively. If the whole area is now lowered (without rotation) to a depth h, find the distance through which the centre of pressure is raised.