43rd C.C. Exam., 2001

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions.

1. (a) Prove that if $V(F)$ is a finite dimensional vector space then any two bases of V have the same number of elements.

   (b) If $V(F)$ and $U(F)$ be vector spaces over a field F and if $f : V \to U$ be a linear transformation from V on to U and its kerneel is K then prove that $\frac{V}{K} \cong U$.

2. (a) Expand $e^{a \sin^{-1} x}$ by Maclaurin's theorem and find the general term.

   (b) Prove that for any curve $\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$, where ρ is the radius of curvature and $\phi = r\frac{d\theta}{dr}$.

   (c) Show that the envelope of the family of parabolas $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$, under the condition $ab = c^2$ is a hyperbola whose asymptotes coincide with the axes.

3. (a) Prove that $\lim\limits_{x \to 0} \frac{\{(1+x)^n - 1\}}{x} = n$.

   (b) Draw the graph of the function $y = [x] + |1 - x|, -1 \le x \le 3$. Determine the points, if any, at which this function is not differentiable.

   (c) Evaluate : $\lim\limits_{x \to \infty}\left(1 + \frac{a}{x}\right)^x$.

4. (a) Show that $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \pi\left(\frac{1}{2}\pi - 1\right)$.

(b) Find the intrinsic equation of the cardioid $r = a(1 - \cos\theta)$.

(c) Find the area of the region bounded by the curve $y = (x-1)(x-2)(x-3)$, ordinates $x = 0$ & $x = 3$ and the axis of x.

5. (a) Prove that $\int_0^x \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx = \frac{B(m,n)}{a^n(1+a)^m}$.

(b) Find the volume of the figure cut from the shere $x + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

6. (a) Solve : $\frac{d^2y}{dx^2} + y = x\sin^2 x\cos^2 x$.

(b) Find the solution of $p^2 + 2xp - y = 0$ and explain its singular solution. Is it true that the cuspidal locus can coincide with the envelope ? Discuss.

(c) Solve : $(2x^2y - 3y^4)\frac{dy}{dx} + (3x^3 + 2xy^3) = 0$.

7. (a) Find the equations to the normals at the ends of the laterarecta of an ellipse and prove that each passes through an end of the minor axis if $e^4 + e^2 = 1$.

(b) If e and e' be the eccentricities of a hyperbola and is conjugate, prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

8. (a) Prove that six normals can be drawn to an ellipsoid from a given point.

(b) Show that the necessary and sufficient condition that a curve lies on a sphere is $\frac{\rho}{\tau} + \frac{d}{dx}\left(\frac{\rho'}{\sigma}\right) = 0$.

9. (a) A covariant tensor has components xy, $2y - z^2$ and xy in rectangular co-ordinats. Find its covariant components in spherical co-ordinates.

(b) Show that any inner product of the tensor A_r^p and B_t^{qs} is a tensor of rank three.

(c) Derive transformation laws for the Christoffel symbols of first kind.

10. (a) Show that :

$[a \times p, b \times q, c \times r] + [a \times q, b \times r, c \times p] + [a \times r, b \times p, c \times q] = 0$

(b) Evaluate : $\nabla \cdot (A \times r)\, \nabla \times A = 0.$

(c) Verify Stokes theorem for $A = (2x - y)i - yz^2 j - y^2 zk$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

11. (a) The extremities of a heavy string, of length $2l$ and weight $2lw$ are attached to two small rings which can slide on a fixed horizontal wire. Each of these rings is acted on by horizontal force equal to lw. Show that the distance between the rings is

$2l \log\,(1 + \sqrt{2})$.

(b) A man is cycling at the rate of 6 miles per hour up a hill whose slope is 1 in 20. If the weight of the man and the cycle be 200 pounds, prove that he must be working at least at the rate of 0.16 H.P.

12. (a) A particle moves with central acceleration $\mu\left(r + \frac{a^4}{r^3}\right)$ being projected from an apse at a distance with a velocity $2\sqrt{\mu a^2}$, show that it describes the curve $r^2(2 + \cos\sqrt{3}\,\theta) = 3a^2$.

(b) It a quadrilateral area be entirely immersed in water, and α, β, γ & δ be depths of its four corners, and h that of its centre of gravity; show that the depth of its centre of pressure is

$$\frac{1}{2}(\alpha + \beta + \gamma + \delta) - \frac{1}{6h}(\beta\gamma + \gamma\alpha + \alpha\beta + \alpha\delta + \beta\delta + \gamma\delta).$$

MATHEMATICS, PAPER – II

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions, selecting at least two question from each Section.

SECTION–A

1. (a) If the group G is abelian and N is any subgroup of G, prove that G/N is abelian.