42nd C.C. Exam., 1999

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 200**

*Answer any **five** questions.*

1. (a) If U and W are subspaces of a finite dimensional vector space V, then prove that :

$$\dim (U + W) = \dim U + \dim U \cap W$$

(b) If $[S]$ denotes the subspace spanned by a non-empty subset S of a vector space V over F, then show that :

$$[S] = \{a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n : a_1, a_2 ....., a_n \in F, \alpha_1, ..., \alpha_n \in S ; n \in N\}$$

2. (a) Show that a set S of non-zero vectors $\{a_1, \alpha_2, ..., \alpha_n\}$ is linearly depended if and only if one of the $a_m \in S, m > 1$ is a linear combination of the preceding vectors.

(b) Show that the only real value of λ for which of the following equations have non-zero solution is 6 :
$x + 2y + 3z = \lambda x$; $3x + y + 2z = \lambda y$ and $2x + 3y + z = \lambda z$.

(c) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

3. (a) Find the value of $\lim_{x \to \alpha} \dfrac{\cos x - \cos \alpha}{\cot x - \cot \alpha}$.

(b) The normal at the point $P(at^2, 2at)$ of the parabole $Y^2 = 4ax$ cuts the same parabole at Q. Find the minimum length of PQ.

(c) Show that the asymptotes of
$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$
from a square, through two of whose angular points the curve passes.

4. (a) Find the value of $\int_{\pi/4}^{3\pi/4} \dfrac{X}{1 + \sin x} dx$.

(b) Show that $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{x}}{2}$.

(c) Find the volume of that part of the cylinder $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1$ which lies between the planes $y = 0$ and $y = mx$.

5. (a) Find the locus of midpoints of the portion of lines $x \cos\alpha + y \sin\alpha = p$ (p is constant) which is intercepted between the axes.

(b) Show that the equation $bx^2 - 2hxy + a^2y = 0$ represents a pair of straight lines which are at right angles of the pair given by the equation $ax^2 + 2hxy + by^2 = 0$.

(c) If the angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°, then show that the locus of P is a hyperbola.

6. (a) Prove that the locus of intersection of normals at the ends of conjugate diameters of an ellipse is the curve.

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

(b) If a circle and a rectangular hyperbola $xy=c^2$ meet in four points 't_1', 't_2', 't_3' and 't_4' then prove that $t_1, t_2, t_3, t_4=1$.

7. (a) For the curve $\gamma=a(3u-u^3, 3u^2, 3u+u^3)$, prove that

[illegible] $\dfrac{1}{3a(1+u^2)^2}$

(b) Find the involute and evolute of the circular helix $\gamma=(a\cos\theta, a\sin\theta, a\theta\cot\alpha)$.

8. (a) Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $x^2+by^2+cz^2=1$.

(b) Prove that the normals from (α, β, γ) to the paraboloid.

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ lie on the come $\dfrac{\alpha}{x-\alpha}-\dfrac{\beta}{y-\beta}+\dfrac{a^2-b^2}{z-\gamma}=0.$

9. (a) Solve : $(1+xy)\,y\,dx+(1-xy)\,x\,dy=0$.

(b) Solve : $y-2px=f(xp^2)$.

(c) Solve : $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=xe^x\sin x$.

10. (a) Solve : $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y$.

(b) Solve : $(D^2+4)\,y=\cos^2 x-\sin^2 x$.

(c) Investigate $xp^2-(x-a)^2=0$ for singular solution.

11. (a) Prove $\nabla\times(\nabla\times A)=-\nabla^2A+\nabla(\nabla\cdot A)$, where the symbols have usual meaning.

(b) If $\phi(x, y, z)=zy^2z$ and $A=xzi-xy^2j+yz^2k$, then find $\dfrac{\partial^3}{\partial x^2\partial z}(\phi A)$ at $(2, -1, 1)$.

(c) Verify Stoke's theorem for $A=(2x-y)\,i-yz^2j-y^2zk$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

12. (a) Show that $\left\{ \begin{matrix} i \\ ij \end{matrix} \right\} = \dfrac{\partial \log \sqrt{g}}{\partial x^i} = \dfrac{\partial \log \sqrt{-g}}{\partial x^j}$.

(b) Prove that the covariant derivatives of the tensors g^{ij}, g_{ij} and ∂^i_j all vanish identically.

13. (a) A particle moves under a force $m\mu\{3au^4 - 2(a^2 - b^2)u^5\}$. $a > b$ and is projected from an apse at a distance $a + b$ with velocity $\dfrac{\sqrt{\mu}}{a+b}$, show that its orbit is $\gamma = a + b\cos\theta$.

(b) Two equal forces act one along each of the straight lines

$$\frac{x \pm a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{\mp b\cos\theta} = \frac{z}{c};$$

show that their central axis must, for all values of θ, line on the surface $y\left(\dfrac{x}{z} + \dfrac{z}{x}\right) = b\left(\dfrac{a}{c} + \dfrac{c}{a}\right)$.

14. (a) A smooth parabolic wire is fixed with its axis vertical and vertex downwards and it is placed in a uniform rod of length $2l$ with its ends resting on wire. Show that, fore equilibrium, the rod is either horizontal or makes with the horizontal an angle θ given by $\cos^2\theta = \dfrac{2a}{l}$, $4a$ being the latus rectum of the parabola.

(b) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is $\mu \log \dfrac{1 + \sqrt{1 + \mu^2}}{\mu}$, where μ is the coefficient of friction.

15. (a) A planet of mass M and periodic time T, when at its greatest distance from the sun comes into collision with a meter of mass m moving in the same orbit in the opposite direction with

Scanned by CamScanner