(c) A particle describes an ellipse under a central force $\mu/(\text{distance})^2$ towards the focus. If it was projected with velocity V from a point distant R from the centre of force, show that its periodic time is $\dfrac{2\pi}{\sqrt{\mu}}\left[\dfrac{2}{R}-\dfrac{V^2}{\mu}\right]^{-3/2}$.

12. (a) A quadrilateral $ABCD$ has the side CD in the surface of a liquid. Its sides BC and AD are vertical and of lengths a and b respectively. Prove that the depth of the centre of pressure is

$$\frac{1}{2}\frac{(a^2+b^2)(a+b)}{a^2+ab+b^2}.$$

(b) A hemispherical bowl is filled with water, and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface. If α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical is $\tan^{-1}\left(\dfrac{\sin\alpha}{\alpha}\right)$.

(c) A heavy uniform rod of thin uniform cross-section and of length is movable in a vertical plane about a hinge at one of its extremities, situated at a depth h $(< l)$ below the surface of liquid of density ρ. If the density of the rod be σ $(< \rho)$, show that the inclination to the vertical at which the rod may rest, is $\cos^{-1}\left(\dfrac{h}{l}\sqrt{\dfrac{\rho}{\sigma}}\right)$.

MATHEMATICS, PAPER – II

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions, selecting at least two question from each Section.

SECTION–A

1. (a) Define automorphism, inner automorphism and outer automorphism of a group. Give an example of an outer automorphism. Let G be a group and $I(G)$ be the group of all inner automorphisms of a group G. Prove that $I(G) \cong G/Z$ where Z is the centre of G.

(b) Let G be a finite group and p be a prime number such that $p/o(G)$. Prove that the number of Sylow p-subgroups in G is of the form $1 + kp$ where k is a non-negative integer. Prove that a group of order 72 is not simple.

2. (a) Define unique factorization domain. Give an example of an integral domain which is not a unique factorization domain. If R is a unique factorization domian, prove that $R[x]$ is also a unique factorization domain.

(b) Define Euclidean ring. Show that the ring of Gaussian integers is Euclidean ring and find all the units in the ring.

3. (a) Define open set in a metric space and show that the collection of all open subsets of a metric space X forms a topology on X.

(b) Define uniform continuity. Give an example of a continuous function which is not uniformly continuous. Prove that every continuous function on a compact metric space (X, d) into a metric space (Y, ρ) is uniformly continuous.

4. (a) Define Rieman-Stieltyes integral. If f is monotonic on $[a, b]$ and is continuous on $[a, b]$, then show that $f \in R(\alpha)$.

(b) Prove that a function $f = (f_1, f_2, \ldots, f_m)$ on IR^n into IR^m is continuously differentiable if and only if the partial derivatives $\delta f_k/\delta x_j$ exist and are continuous for $1 \leq k \leq m$ and $1 \leq j \leq n$.

5. (a) State and prove Riemann's theorem on the rearrangement of terms of a conditionally convergent series.

(b) Show that the series $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \ldots$ is uniformly convergent for all values of x, but not absolutely convergent.

6. (a) State and prove Weierstrass M-test for uniform convergence. Give an example of a series which is convergent pointwise but not uniformly convergent.

(b) Find maxima and minima of the function f given by

$$f(x, y) = x^3 + y^3 + 12xy - 63x - 63y.$$

7. (a) Define differentiability of a function of a complex variable. Prove that Cauchy-Riemann equations are necessary for differentiability but they are not sufficient.

(b) Using contour integration evaluate the following :

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx.$$

8. (a) Apply Charpit's method to find the complete integral of $z^2(p^2z^2+q^2)=1$.

(b) Solve : $(D^2-2DD'-15D'^2)z=12xy$ where $D\equiv\frac{\delta}{\delta x}$ and $D'\equiv\frac{\delta}{\delta y}$.

SECTION–B

9. (a) Four letters addressed to different persons are put at random into four envelopes bearing their addresses. Obtain the probability that only one letter goes into the correct envelope.

(b) If n_1, n_2 are the sizes $\bar{x}_1, \bar{x}_2$ the means and σ_1, σ_2 the standard deviations of two series, then prove that the standard deviation of the combined series is given by

$$\sigma^2=\frac{1}{n_1+n_2}[n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)] \text{ where}$$

$$d_1=-\bar{x}_1-\bar{x},\ d_2=\bar{x}_2 \text{ and } \bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2},$$ is mean of the combined series.

10. (a) A uniform sphere rolls down an inclined plane rough enough to prevent any sliding. Determine the motion.

(b) A particle Q moves on a smooth horizontal circular wire of radius a which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the $\angle QCO=\theta$, show that $a\theta+\omega(a-c\cos\theta)=c\,\omega^2\sin\theta$, where ω is the angular velocity of the wire.

11. (a) Show that $u=-\frac{2xyz}{(x^2+y^2)^2}, v=\frac{(x^2-y^2)z}{(x^2+y^2)^2}, w=\frac{y}{x^2+y^2}$ are the velocity components of a possible liquid motion. Is this motion irrotational ?

(b) Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if be the density and v be the velocity at a distance x from a fixed point at the time t, then

$$\frac{\delta^2\rho}{\delta t^2}=\frac{\delta^2}{\delta x^2}[\rho(v^2+k)]$$

12. (a) Use Regula Falsi method to find the real root of the equation $x^3 - x^2 - 2 = 0$, correct to three decimal places.

(b) Show that the generalised Newton's formula

$x_{n+1} = x_{n-2}\left(\frac{f(x_n)}{f'(x_n)}\right)$ gives quadratic convergence when the equation $f(x) = 0$ has a pair of duble roots in the neighbourhood of $x = x_n$.

13. (a) By means of Newton's divided difference formula find the value of $f(8)$ and $f(15)$ from the following table :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(b) Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ at $x = 1$ and $\frac{dy}{dx} = 3x + y^2$.

14. (a) Solve the following linear programming problem by simplex method :

Maximize $2x_1 + x_2 + 3x^3$ subject to the contraints $x_1 + x_2 + 2x_3 \le 5$; $2x_1 + 3x_2 + 4x_3 = 12$; $x_1, x_2, x_3 \ge 0$.

(b) Discuss in brief 'Duality' in linear programming. How can the optimal solution of primal be obtained from the optimum solution of the dual ?

15. (a) Solve the transportation problem for which the cost, origin availabilities and destination requirements are given below :

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	175 (Total)

(b) Use dynamic programming method to minimize

$$z = y_1^2 + y_2^2 + y_3^2$$

subject to the constraints

$$y_1 + y_2 + y_3 \ge 15; \quad y_1, y_2, y_3 \ge 0.$$

16. (a) Use the Kuhn-Tucker conditions to solve the following non-linear programming problem :

Maximize $z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

subject to the constraints : $2x_1 + 5x_2 \leq 98$; $x_1, x_2 \geq 0$

(b) We have five jobs, each of which must go through machines A, B and C in the order ABC. Processing times are given in the following table :

Job	Processing times		
i	A_i	B_i	C_i
1	8	5	4
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

Determine a sequence for the five jobs that will minimize the elapsed time T.