41st C.C. Exam., 1997

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions.

1. (a) If W is a subspace of a finite dimensional vector space V, then prove that $\dim V/W = \dim V - \dim W$.

(b) Find the eigenvalues and eigenvectors of the matrix :

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

(c) State and prove Caley-Hamilton theorem and use it to find the inverse of $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$.

2. (a) Let L : $V \to W$ be a linear transformation. Prove that Im L (image of L) is a subsapce of W and Ker L (kernel of L) is a subspace of V.

(b) (i) If $H = \begin{bmatrix} 3 & 5+2i & -3 \\ 5-2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$

then show that iH is a skew-Hermitian matrix.

(ii) If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

then show that $(I - N)(I + N)^{-1}$ is unitary.

(c) Find the rank of the matrix : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & a \\ 2 & 2a-2 & -a-2 & 3a-1 \\ 3 & a+1 & -3 & 2a-1 \end{bmatrix}$.

3. (a) (i) For what values of p is the function :

$$f(x) = \begin{cases} x^p \sin(1/x) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

both continuous and differentiable at $x = 0$?

(ii) Evaluate : $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$.

(b) State and prove Lagrange's mean-value theorem. Also give its geometrical significance.

(c) (i) If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y} \right)$, show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \tan u$.

(ii) Find the curve on which the points of intersection of the curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie.

(iii) Determine the points where the function $f(x, y) = x^3 + y^3 - 3axy$ has a maximum or a minimum.

4. (a) (i) Prove that : $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}$ (n = integer > 1).

(ii) Evaluate : $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

(b) Find the volume in the positive octant of the ellipsoid :

$x^2/a^1 + y^1/b^2 + z^2/c^2 = 1$

(c) (i) Define the beta function $B(m, n)$ and the gamma function Γx and show that $B(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma m + n}$.

(ii) Find the centre of gravity of a semi-circle of radius a.

5. (a) Prove that the general equation :
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines if $h^2 = ab$ and $bg^2 = af^2$. Prove also that the distance between them is $2\sqrt{\{(g^2 - ac)/a(a+b)\}}$.

(b) Show that the equation of the circle passing through the co-normal points of the parabola $y^2 = 4ax$ is $x^2 + y^2 - (h+2a)x - \frac{1}{2}ky = 0$, (h, k) being the co-ordinates of the point where the normals meet.

(c) Show that the lengths of the semi-axes of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by $(ab - h^2)^3 r^4 + \Delta(a+b)(ab - h^2) r^2 + \Delta^2 = 0$ where Δ is the discriminant of the quation of the conic.

6. (a) A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C; Find the locus of the centre of the sphere $OABC$.

(b) Tangent planes are drawn to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ through the point. Prove that the perpendiculars to them from the origin lie on the cone :

$$(\alpha x + \beta y + \gamma z)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

(c) Establish Frenet formulae and show that $[\bar{r}', \bar{r}'', \bar{r}'''] = k^2\tau$, where k is the curvature and τ is the torsion of the space curve $\bar{r}(s)$ and dashes denote differentiations with respect to the arc-length s.

7. (a) Find the differential equation of a system of confocal and coaxal parabolas in a plane.

(b) (i) Solve : $(2x - 10y^3)\,dy + y\,dx = 0$

(ii) Solve : $\cos(x + y)\,dy - dx = 0$

(c) Find the integrating factor and solve :

$$(3xy + 2y^3)\,dx + (4x^2 + 6xy^2)\,dy = 0$$

given that $y = 2$ when $x = 1$.

8. (a) Solve : $d^4y/dx^4 + m^4y = 0$

(b) Solve : $d^2y/dx^2 + y = c^{-x} + \cos x + x^3 + e^x \sin x$.

(c) A particle falls under constant force of gravity in a resisting medium whose resistance varies as the square of the velocity. Find the velocity and the distance of the particle at time t.

9. (a) (i) For a unit vector $\hat{a}(t)$, show that $\left|\hat{a} \times \frac{d\hat{a}}{dt}\right| = \left|\frac{d\hat{a}}{dt}\right|$.

(ii) Find the unit normal vector of the surface

$x^4 - 3xyz + z^2 + 1 = 0$ at the point (1, 1, 1).

(b) State Stoke's theorem and verify it for the vector function.

$\overline{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface $x + y + z = 1$ lying in the first octant.

(c) Define Christoffel symbols and show that these are not tensors.

10. (a) Enunciate the principle of virtual work. Four equal jointed rods, each of length a, are hung from an angular point, which is connected by an elastic string with the opposite point. It the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the wieght of a rod, show that unstretched length of the string is $a\sqrt{2/3}$.

(b) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable is $\sqrt{3}$ times the radius of the hemisphere.

(c) A uniform chain of length 1, is to be suspended from two points, A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be $\frac{1}{\sqrt{n^2-1}} \log_e[n + \sqrt{n^2 - 1}]$.

11. (a) A particle moves under gravity in a vertical circle, sliding down the convex side of a smooth circular arc. If its initial velocity is that due to a fall to the starting point from a height h above the centre, show that it will fly off the circle when of a height $2h/3$ above the centre.

(b) A body of mass $(m_1 + m_2)$ is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that if after explosion the parts move in the same line as before, their relative speed is $\sqrt{2E(m_1 + m_2)/(m_1 m_2)}$.