12. (a) A thin hollow cone, with a base, floats completely immersed in water where it is placed. Show that the vertical angle of the cone is $2\sin^{-1}\left(\frac{1}{3}\right)$.

(b) A parallelogram has its corners at depths, h_1, h_2, h_3, h_4, below the surface of a liquid and its centre at a depth h, show that the depth of its centre of pressure is

$$\frac{h_1^2 + h_2^2 + h_3^2 + h_4^2 + 8h^2}{12th}.$$

(c) A zone whose vertical angle is 2α, has its lowest generator horizontal and is filled with a liquid. Prove that the resultant pressure on the curved surface is

$$w = \sqrt{(1 + 15\sin\alpha^2)}$$

where w is the weight of the liquid contained.

MATHEMATICS, PAPER – II

Time : 3 Hours] **[Max. Marks : 200**

Answer any five questions, selecting at least two question from each Section.

SECTION–A

1. (a) Define 'group'. Show that the set Zp of non-zero residue classes module a prime number p, is a multiplicative group.

(b) Let G be the multiplicative group of all 2×2 real matrices with non-zero determinant. Show that $H = \{A \in G \text{ 1 } \det A = 1\}$; is a normal subgroup of G.

(c) Let G be a multiplicative group. Show that $f : G \to G$ defined by $f(x) = x^{-1}$, is a group isomorphism if G is abelian.

2. (a) State and prove Sylow's first theorem.

(b) Prove that every group is isomorphic to a permutation group.

(c) Prove that a division ring R has no ideal different from $\{O\}$ and R.

3. (a) Prove that every Euclidean domain, is a principal ideal domain.

(b) If F is a field, prove that $F[x]$ is a unique factorization domain.

(c) Let F and K be finite fields such that FCK. If F has q elements, show that K ahs q^n elements where $n = [K : F]$.

4. (a) Let $< x, d >$ be a metric space. For $a, b \in x$, define $e(a, b) = \min(1, d)(a, b)$. Show that $< x, e >$ is also a metric space.

(b) Prove that in a metric space, every convergent sequence is a Cauchy sequence. Is the converse also true ? Give an example in support of your answer.

(c) Show that every compact metric space is complete.

5. (a) Prove that a continuous image of a compact space is compact.

(b) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$.

(c) Show that the maximum values of $f(x, y) = 2(x-y)^2 - x^4 - y^4$ are $f(\sqrt{2}, -\sqrt{2})$ and $f(-\sqrt{2}, \sqrt{2})$.

6. (a) If $(u, (x), n = 1, 2, 3, ...$ are continuous in $[a, b]$ and if $\Sigma u, (x)$ converges uniformly to the sum $S(x)$ in $[a, b]$ prove that

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty}\int_a^a u_n(x)\,dx.$$

(b) If f is continuous over $[a, b]$ and g is of bounded variation over $[a, b]$, prove that f is Riemann Steilties integrable with respect to g over $[a, b]$.

If $\int_a^{\infty}|f(x)|\,dx$ converges, prove that $\int_a^{\infty} f(x)\,dx$ converges.

7. (a) If $f(z)$ is analytic at z_0, prove that it is continuous at z_0. Show by an example that the converse is not always true.

(b) Let $f(z)$ be analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve C, prove that $\oint_c f(z)\,dz = 0$.

(c) Evaluate : $\oint_C \dfrac{\sin(\pi z^2)+\cos(\pi z^2)\,dz}{(z-1)(z-2)}$.

where C is the circle $|z| = 3$.

8. (a) Expand $1/z^2$ in a Taylor series about the point $z = 1$.

(b) Evaluate $\int_0^{\infty}\dfrac{dx}{1+x^2}$ by the method of contour integration.,

(c) State and prove the Residue Theorem. Deduce that the sum of residues of $f(z)$ at all its singularities including infinity, is zero.

SECTION–B

9. (a) Apply charpit's method to find complete integral of $2xz - px^2 - 2qxy + pq = 0$.

(b) Eliminate the function f from $z = y^2 + 2f(1/x + \log y)$.

(c) If $D \equiv \frac{\delta}{\delta x}, D' \frac{\delta}{\delta y}$, solve $(D^2 - a^2 D'^2)z = x$.

10. (a) Define holonomic and non-holonomic dynamical systems. Derive Lagrange's equations for the non-holonomic system.

(b) Deduce the principle of energy from Lagrange's equations.

(c) A uniform beam of mass m and length $2l$, is horizontal and can turn freely about its centre which is fixed. A particle of mass m' and moving with vertical velocity u, hits the beam at one end. If the coefficient of restitution be e, show that the angular velocity of the beam just after the impact, is

$$\frac{3m'(1+e)u}{(m+3m')l}.$$

11. (a) Find the moment of inertia of a uniform circular cylinder about an axis through its centre of gravity, perpendicular to its axis.

(b) State D'Alembert's principle and deduce general equations of motion of a rigid body by using it.

(c) For a body moving in two dimensions, express the kinetic energy in terms of the motion of centre of inertia.

12. (a) What physical significance is implied in the equation of continuity ? Express the equation of continuity in the form

$$\frac{\delta p}{\delta t} + \Delta \cdot v = 0.$$

(b) State and prove Euler's hydrodynamical equations. Establish Bernoulli's theorem that in the case of steaby motion of a homogeneous inelastic fluid $(p/p_1) + \frac{1}{2}q^2 + V = K$, where K is a constant along a stream line but varie from stream line to stream line.

(c) Two sources, each of strength m, placed at the points $(+a, 0)$ and a sink of strength $2m$ is placed at the origin. Show that the stream lines are the curves, $(x^2 + y^1)^2 = a^2 (x^2 - y^2 + \lambda x_0)$, where λ is a variable parameter.

13. (a) Describe Newton-Raphson method for finding the real roots of a polynomial equation. Give the condition for its convergence.

(b) Solve $x^3 - 9x + 1 = 0$ for the root lying between 2 and 4 by the method of false position.

14. (a) If yx is 4 polynomial in x of third degree, find on expression for $\int_0^1 y_2\, dx$ in terms of y_0, y_1, y_2 and y_3. Use this result to show that $\int_0^1 y_2\, dx = 1, 2\frac{1}{2}$ x $[-y_0 + 13y_1 + 13y_2 + y_3]$

(b) Given the differential equation $\frac{dy}{dx} = \frac{-(x-a)\,y}{b_0 + b_1 x + b_2 x^2}$, obtain the values of a, b, b_1 and b_2 in terms of moments.

15. Explain the difference between a transportation problem and an assignment problem. Solve the following transportation problem.

Destination

Source	1	2	3	4	Supply
1	21	16	15	3	11
2	17	18	14	23	13
3	32	27	18	14	19
Demand	6	10	12	15	43

16. (a) What is meant by queue ? Give the essential characteristics of the queueing process.

(b) If the number of arrivals in some time interval follows a poisson distribution, show that the time interval between two consecutive arrivals is exponential.

17. (a) Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools choosen at random, exactly two will be defective by using (i) the binomial distribution, (ii) the poisson approximation to the binomial distribution.

(b) Define (i) line of regression, (ii) regression coefficient. Find the equations to the lines of regression and show that the coefficient of correlation is the geometric mean of coefficients of regression.

18. (a) Define random variable and its mathematical expectation. What is the covariance of two random variables ?

(b) State and prove the weak law of large numbers.

(c) Write the probability density function of two parameter gamma distribution and find its mean and variance.