40th C.C. Exam., 1995

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 200**

*Answer any **five** questions.*

1. (a) If U and W are subspaces of a finite-dimensional vector space VF, then prove that

$$\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$$

(b) Suppose that x and y are vectors and U is a subspace of a vector space V_f. Let W be the subspace spanned by U and x and let Y be the subspace spanned by U and y. Prove that if $y \Sigma W - U$, then $x \Sigma Y$.

(c) Show that the vectors (Σ_1, Σ_2) and (η_1, η_2) in C^2 are linearly dependent if and only if $\Sigma_1, \eta_2 = \Sigma_2 \eta_1$.

2. (a) Define a linear transformation from a vector space U into a vector space V. Show that the set $L\ (V \cdot V')$ of all linear transformations from a vector space V to a vector space V', is a vector space under suitably defined addition and scalar multiplication.

(b) For any linear transformation $T : V \rightarrow V$ prove the following statements :

(i) If $v_1, v_2, \ldots\ldots, v_n$ are linearly dependent vectors in V, then $T(v_1), T(v_2) \ldots., T(v_n)$ are linearly dependent vectors in V.

(ii) $\gamma (T) \leq \min (\dim V, \dim V')$ where γT denotes the rank of T and dim V and dim V' denote the dimention of V and V' respectively.

(c) Find a set of three orthonormal eigenvectors for the matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{bmatrix}.$$

3. (a) Assuming $f''(x)$ continuous in (a, b), show that

$$f(c)-f(a)\,\frac{b-c}{b-a}-f(b)\,\frac{c-a}{b-a}=\frac{1}{2}(c-a)\,(c-b)f''(\Sigma)$$

Where C and Σ both lie in (a, b).

(b) Prove that the area of the triangle formed by the tangents at any point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its axes, is a minimum for the point $\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$.

(c) Evaluate him $\lim\limits_{x\to 0}\dfrac{(1+x)\frac{1}{x}-e+\frac{lex}{2}}{x^2}$.

4. (a) Show that $\int_0^{\pi} x\log\sin x\,dx=\frac{1}{2}\pi^2\log\frac{1}{2}$.

(b) The loop of the curve $2ay^2=x(x-a)^2$ revolves about x-axis, find the volume of the solid so generated.

(c) Prove that $\int_0^{\infty}\frac{\tan^{-1}(ax)\,dx}{x(1+x^2)}=\frac{1}{2}\pi\log(1+a)$ if $a\geq 0$.

5. (a) Prove that the locus of the poles of normal chords with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is the curve

$$a^6y^2-b^6x^2=(a^2+b^2)^2x^2y^2.$$

(b) A sphere of constant radius passes through the origin 0 and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from 0 to the plane ABC.

(c) Prove that the property $\frac{K}{t}$ = constant characterizes a helix, where K denotes the curvature and t denotes the tersion.

6. (a) Show that the equation

$$(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$$

represents a family of hyperbolas having as asymptotes the lines $x + y = \theta$ and $2x + y + 1 = 0$.

(b) Solve : $\left(xy^2 - \frac{1}{e^3}\right)dx - x^2y\,dy = 0$

(c) Solve : $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

7. (a) Find the value of y which satisfies the differential equation :

$$\frac{d^2y}{dx^2} - 9\frac{dy}{xx} + 20 = 0$$

given that $y = 0$ and $\frac{dy}{dx} = 1$, when $x = 0$.

(b) Solve : $\frac{d^2y}{dx^2} + n^2y = \sec nx$.

(c) Solve : $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 + \sin x^2)$.

8. (a) Define the divergence of a vector point function and prove that div $(sU) = U$. grad $s + s$ div U.

(b) A particle moves along the curve $x - t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$.

(c) Use Gauss divergence theorem to prove that

$$\iint_S \left[xz^2\,dy\,dz + (x^2y - z^2)\,dz\,dx + (2xy + y^2z)\,dy\,dx\right] = \frac{2}{3}\pi a^5$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$.

9. Solve : (a) $D^2 + 2D + 1y$ ..... $x \cos x$

(b) $x^2 \dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} - 3y x^2 \log x$

(c) $\sin x \dfrac{dy}{dx}$..... $\cos y(1 - x\cos y)$.

10. (a) In a smooth hemispherical cup is placed a heavy rod, equal in length to the radius of the cup, the centre of gravity of the rod being one-third of its length from one end; show that the angle made by the rod with the vertical is $\tan^{-1}(3\sqrt{3})$.

(b) A given length $2s$ of a uniform chain case to be hung between two points at the same level and the tension has not to exceed the weight of a length le of the chain. Show that the greatest span is $\sqrt{le^2 - s^2} \log \dfrac{le + s}{le - s}$.

(c) A rod, resting on a rough inclined plane, whose inclination α to the horizon is greater than the angle of friction λ, is free to turn about one of the ends, which is attached to the plane; show that, for equilibrium, the greatest possible inclination of the rod to the line of the greatest slope is $\sin^{-1}(\tan\lambda \cot\alpha)$.

11. (a) The velocity of a train increases at a constant rate f_1 from rest to v, then remains constant for an interval and finally decreases to zero at the constant rate f_2. If x is the total distance described prove that the total time taken is $\dfrac{x}{v} + \dfrac{v}{2}\left(\dfrac{1}{f_1} + \dfrac{1}{f_2}\right)$.

(b) A particle of mass m, is projected vertically under gravity, the resistance of the air being mk times the velocity, show that the greatest height attained by the particle is $\dfrac{v^2}{g}[\lambda - \log(1 + \lambda)]$. where v is the terminal velocity of the particle and λv is the initial vertical velocity.

(c) Find the law of force towards the pole under which the curve $r = a\, e\theta \cot\alpha$ is described.