39th C.C. Exam., 1993

MATHEMATICS, PAPER – I

Time : 3 Hours] **[Max. Marks : 100**

Answer any five questions.

1. (a) Define a basis for a vector space. If $v_1, v_2, \ldots\ldots, v_n$, is a basis of a vector space V over F and if $w_1, w_2, \ldots\ldots, w_n$, in V are linearly independent over F, then prove that $M \le n$.

   (b) Under what condition on the scalar Σ do the vectors (1, 1, 1) and $(1, \Sigma, \Sigma^2)$ from a basis for C^3 ?

2. (a) Define the rank and nullity of a linear transformation from a finite dimensional vector space in to another vector space.
   If $T : V_1 \to V_2$ is a linear transformation from a vector space V_1 into a vector space V_2.
   Prove that : $\dim V_1 = R(T) + N(T)$
   Where $R(T)$, $N(T)$ are rank and nullity of T.

   (b) Find a non-singular linear transformation which reduces

   $$3x^2 + 3y^2 + 3z^2 - 2yz + 2zx + 2xy, 2yz - 2zx + 4xy$$

   simultaneously to $X^2 + Y^2 + Z^2$

   $$\alpha_1 X^2 - \alpha_2 Y^2 - \alpha_3 Z^2.$$

   (c) Let A and B be square matrices of the same order n. If A and B are similar, then show that they have the same eigen values.

3. (a) If $f(x)$, $\phi(x)$ and $\mu(x)$ are there, functions derivable in an interval (a, b), show that there exists a point Σ in (a, b) such that

   $$\begin{vmatrix} f'(\Sigma) & \phi'(\Sigma)\mu' & (\Sigma) \\ f'(a) & \phi(a)\mu & (a) \\ f'(b) & \phi(b)\mu & (b) \end{vmatrix} = 0.$$

   (b) If $u = \sin^{-1} \dfrac{x+y}{\sqrt{x} + \sqrt{y}}$ the show that

   $$x^2 \frac{\delta^2 u}{\delta x^2} + 2xy \frac{\delta^2 u}{\delta x \, \delta y} + y^2 = \frac{\delta^2 u}{\delta y^2} + \frac{-\sin u \cos 2u}{4\cos^2 u}$$

(c) A cone is circumscribed to a sphere of radius r show that when the volume of the cone is a minimum, is attitude is $4r$ and its semi vertical angle is $\sin^{-1}\frac{1}{3}$.

4. (a) Show that $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin x} = \frac{\pi^2}{2ab}$.

(b) Prove that $\beta(m, n) = \frac{|(m)|(n)}{l(m+n)}$.

(c) Find the centre of gravity of a plane lamina of uniform density in the form of a quadrant of an ellipse.

5. (a) Show that tangents at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

(b) A sphere of constant radius μ passes through the origin and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.

(c) Find $f(\theta)$, so that $x = a\cos\theta$, $y = a\sin\theta$, $z = f(\theta)$, determines a plane curve.

6. Solve the following differential equations :

(a) $y(2xy + e^x)\,dx - e^x\,dy = 0$

(b) $(x+2y)(dx-dy) = dx + dy$

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)^x \frac{dy}{dx}$$

7. (a) Solve : $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = e^{3x}$.

(b) Solve : $\frac{d^2y}{dx^3} - \frac{d^2y}{dx^2} - \frac{6dy}{dx} = 1 + x^2$.

(c) Solve : $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{2x}\sin 2x$.

8. (a) Show that the necessary and sufficient condition for the vector $v(t)$ to have a constant direction is $v \times \frac{dv}{dt} = 0$.

(b) Prove that div $(f \times g) = g$. Curl $f - f$, Curl g.

(c) Verify Stoke's theorem for the function $F = xi + z^2 j + y^2 k$ over the plane surface $x + y + z = 1$ lying in the first octant.

9. (a) Obtain the laws of transformation of a tensor of the type (1, 1) and show why it is called a contravarint tensor of order 1 and covariant tensor of order 1.

(b) Find whether or not covariant derivation preserves the Skew symmetry or symmetry of a tensor.

(c) Let $\overline{T}_{ij}$ and T_{ij} be the components of two symmetric

$$\overline{T}_{ij} T_{ki} - \overline{T}_{ki} T_{ij} + T_{ij} T_{jk} - \overline{T}_{jk} \Delta_{ij} = 0,$$

show that $\overline{T}_{ji} = K\, \Delta_{if}$ for come real number k.

10. (a) The extremities of a heavy string of length 21 and weight $2/w$ are attached to two small rings which can slide on a fixed horizontal wire. Each of these rings is acted on by horizontal force equal to law. Show that the distance apart of the rings is $2/1 \log (1 + \sqrt{2})$.

(b) A bar rests on two pegs, and makes angle α with the horizontal. The centre of gravity is between the pegs, at a distance a, b from them. Prove that for equilibrium $\tan\alpha \leq \frac{v_1 b + v_2 a}{a + b}$.

where v_1, v_2 are the coefficients of friction at the pegs.

(c) Four equal rods each of weight W are jointed together so as to form a rhombus $ABCD$; the whole hands from the fixed point A, and is kept in shape by a light horizontal road BD. A weight W hangs from C, find the thrust in the rod.

11. (a) A particle moves in an elipse under a force which is always directed towards its focus. Determine the law of force and the velocity at any point of its path.

(b) The speed v of the point P which moves in a line is given by the relation $v_2 = a + 2bx - cx_2$, where x is the distance of the point P from a fixed point on the path and a, b and c are