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UPSC MATHS
OPTIONAL COACHING

Real Analysis

Previous year Questions from 2020 to 1992

2021-22

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2020

1. Prove that the sequence (a_n) satisfying the condition $|a_{n+1} - a_n| \le \alpha |a_n - a_{n-1}|\ 0 \le \alpha \le 1$ for all-natural numbers $0 \le \alpha \le 1$ is a Cauchy sequence. **[10 Marks]**
2. Prove that the function $f(x) = \sin x^2$ is not uniformly continuous on the interval $[0, \infty[$ **[15 Marks]**
3. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}, x \ne y$ then show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial u} + y^2 \dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$ **[20 Marks]**
4. Show that $\int\limits_0^{\pi/2} \dfrac{\sin^2 x}{\sin x + \cos x} dx = \dfrac{1}{\sqrt{2}} \log_e (1 + \sqrt{2})$ **[15 Marks]**

2019

5. Show that the function $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x - y} & (x,y) \ne (1,-1) \quad (1,1) \\ 0 & (x,y) = (1,1) \quad (1,-1) \end{cases}$ is continuous and differentiable at $(1,-1)$ **[10 Marks]**
6. Evaluate $\int\limits_0^{\infty} \dfrac{\tan^{-1}(ax)}{x(1+x^2)} dx, a \ge 0, a \ne 1$ **[10 Marks]**
7. Using differentials, find an approximate value of $f(4.1, 4.9)$ where $f(x,y) = (x^3 + x^2 y)^{\frac{1}{2}}$ **[15 Marks]**
8. Discuss the uniform convergence of $f_n(x) = \dfrac{nx}{1 + n^2 x^2}, \forall x \in \mathbb{R}(-\infty, \infty)\ \ n = 1, 2, 3, \ldots$ **[15 Marks]**
9. Find the maximum value of the $f(x,y,z) = x^2 y^2 z^2$ subject to the subsidiary condition. $x^2 + y^2 + z^2 = c^2, (x,y,z \ge 0)$ **[15 Marks]**
10. Discuss the convergence of $\int\limits_1^2 \dfrac{\sqrt{x}}{\ln x} dx$ **[15 Marks]**

2018

11. Prove the inequality: $\dfrac{\pi^2}{9} < \int\limits_{\pi/6}^{\pi/2} \dfrac{x}{\sin x} dx < \dfrac{2\pi^2}{9}$ **[10 Marks]**
12. Find the range of $p(> 0)$ for which the series: $\dfrac{1}{(1+a)^p} - \dfrac{1}{(2+a)^p} + \dfrac{1}{(3+a)^p} - \ldots., a > 0$

    (i) absolutely convergent and (ii) conditionally convergent. **[10 Marks]**
13. Show that if a function f defined on an open interval (a,b) of $\mathbb{R}$ is convex then f is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous. **[15 Marks]**
14. Suppose $\mathbb{R}$ be the set of all real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a function such that the following equations hold for all $x, y \in \mathbb{R}$:

$(i) f(x+y) = f(x) + f(y)$
$(ii) f(xy) = f(x) f(y)$
Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$, or, $f(x) = x$. **[15 Marks]**

2017

15. Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20},\ n = 1,2,3,...$ show that the sequence $x_1, x_2, x_3..$ is convergent. **[10 Marks]**

16. Find the Supremum and the Infimum of $\dfrac{x}{\sin x}$ on the interval $\left(0, \dfrac{\pi}{2}\right]$. **[10 Marks]**

17. Let $f(t) = \int_0^t [x]\, dx$ where $[x]$ denote the largest integer less than or equal to x
    (i) Determine all the real numbers t at which f is differentiable.
    (ii) Determine all the real numbers t at which f is continues but not differentiable. **[15 Marks]**

18. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100. **[20 Marks]**

2016

19. For that the function $f:(0,\infty) \to R$ given by $f(x) = x^2 \sin\dfrac{1}{x}, 0 < x < \infty$ Show that there is a differentiable function $g: R \to R$ that extends f **[10 marks]**

20. Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:
    $x_1 = \dfrac{1}{2}, y_1 = 1, x_n = \sqrt{x_{n-1} y_{n-1}}, n = 2,3,4,... \dfrac{1}{y_n} = \dfrac{1}{2}\left(\dfrac{1}{x_n} + \dfrac{1}{y_{n-1}}\right), n = 2,3,4,...$ and Prove that $x_{n-1} < x_n < y_n < y_{n-1}, n = 2,3,4,...$ and deduce that both the sequence converges to the same limit l where $\dfrac{1}{2} < l < 1$. **[10 marks]**

21. Show that the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n+1}$ conditionally convergent (if you use any theorem (s) to show it then you must give a proof of that theorem(s). **[15 marks]**

22. Find the relative maximum minimum values of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ **[15 marks]**

23. Let $f: R \to R$ be a continuous function such $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are finite. Prove that is uniformly continuous on $\mathbb{R}$ **[15 marks]**

2015

24. Test for convergence $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\dfrac{n}{n^2+1}\right)$ **[10 Marks]**

25. Is the function $f(x)=\begin{cases} \frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n} \\ 0 & x=0 \end{cases}$ Riemann Integrable? If yes, obtain the value of $\int_0^1 f(x)dx$

[15 Marks]

26. Test the series of functions $\sum_{n=1}^{\infty}\frac{nx}{1+n^2x^2}$ for uniform convergence **[15 Marks]**

27. Find the absolute maximum and minimum values of the function $f(x,y)=x^2+3y^2-y$ over the region $x^2+2y^2\le 1$ **[15 Marks]**

2014

28. Test the convergence of the improper integral $\int_1^{\infty}\frac{dx}{x^2\left(1+e^{-x}\right)}$ **[10 Marks]**

29. Integrate $\int_1^0 f\left(x\right)dx,$,where $f\left(x\right)=\begin{cases} 2x\sin\frac{1}{x}\cos\frac{1}{x}, x\in\left[0,1\right] \\ 0 \qquad x=0 \end{cases}$ **[15 Marks]**

30. Obtain $\frac{\partial^2 f\left(0,0\right)}{\partial x\partial y}$ and $\frac{\partial^2 f\left(0,0\right)}{\partial y\partial x}$ for the function $f(x,y)=\begin{cases} \frac{xy\left(3x^2-2y^2\right)}{x^2+y^2}, \left(x,y\right)\ne\left(0,0\right) \\ 0 \qquad ,\left(x,y\right)\ne\left(0,0\right) \end{cases}$

Also, discuss the continuity $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ of at $\left(0,0\right)$ **[15 Marks]**

31. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz=a^3$ by the method of Lagrange multiplies. **[15 Marks]**

2013

32. Let $f\left(x\right)=\begin{cases} \frac{x^2}{2}+4 & \text{if } x\ge 0 \\ \frac{-x^2}{2}+2 & \text{if } x<0 \end{cases}$ Is f Riemann integrable in the interval $\left[-1,2\right]$? Why? Does there exist a function g such that $g'(x)=f(x)$? Justify your answer. **[10 Marks]**

33. Show that the series $\sum_1^{\infty}\frac{(-1)^{n-1}}{n+x^2}$,is uniformly convergent but not absolutely for all real values of x **[13 Marks]**

34. Show that every open subset of R is countable union of disjoint open intervals **[14 Marks]**

35. Let $\left[x\right]$ denote the integer part of the real number x, i.e., if $n\le x<n+1$ where n is an integer, then $\left[x\right]=n$. Is the function $f\left(x\right)=\left[x\right]^2+3$ Riemann integrable in the function in $\left[-1,2\right]$? If not, explain why. If it is integrable, compute $\int_{-1}^{2}\left(\left[x\right]^2+3\right)dx$ **[10 Marks]**

2012

36. Let, $f_n(x) = \begin{cases} 0, & \text{if } x < \dfrac{1}{n+1} \\ \sin\dfrac{\pi}{x}, & \text{if } \dfrac{1}{n+1} \le x \le \dfrac{1}{n} \\ 0, & \text{if } x > \dfrac{1}{n} \end{cases}$, Show that $f_n(x)$ converges to a continuous function but not uniformly. **[12 Marks]**

37. Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent **[12 Marks]**

38. Let $f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y) \ne (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$ Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$. **[15 Marks]**

39. Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ and from the origin, by the method of Lagrange's multipliers. **[15 Marks]**

40. Let $f(x)$ be differentiable on $[0,1]$ such that $f(1)=f(0)=0$ and $\int_0^1 f^2(x)\,dx = 1$. Prove that

$\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$ **[15 Marks]**

41. Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify your answer **[20 Marks]**

2011

42. Let $S=(0,1)$ and f be defined by $f(x)=\frac{1}{x}$ where $0<x\le 1$ (in R). Is f uniformly continuous on S ? Justify your answer. **[12 Marks]**

43. Let $f_n(x) = nx(1-x)^n$, $x \in [0,1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0,1]$ **[15 Marks]**

44. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2+8xy+7y^2=225$ **[15 Marks]**

45. Show that the series for which the sum of first n terms $f_n(x) = \frac{nx}{1+n^2x^2}$, $0 \le x \le 1$ cannot be differentiated term-by-term at $x=0$. What happens at $x \ne 0$? **[15 Marks]**

46. Show that if $S(x) = \sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$, then its derivative $S'(x) = -2x\sum_{n=1}^{\infty}\frac{1}{n^2\left(1+nx^2\right)^2}$, for all x **[20 Marks]**

2010

47. Discuss the convergence of the sequence $\{x_n\}$ where $X_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$ **[12 Marks]**

48. Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$ Show that the sequence converges to $\left(\dfrac{1+\sqrt{17}}{2}\right)$

[12 Marks]

49. Define the function $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Justify your answer.

[15 Marks]

50. Consider the series $\sum\limits_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^2}$. Find the values of x for which it is convergent and also the sum function. Is the converse uniform? Justify your answer. **[15 Marks]**

51. Let $f_n(x) = x^n$ on $-1 < x \leq 1$ for $n = 1, 2, \ldots$. Find the limit function. Is the convergence uniform? Justify your answer. **[15 Marks]**

2009

52. State Roll's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root o $e^x \sin x = 1$ **[2+10=12 Marks]**

53. Let $f(x) \begin{cases} \dfrac{|x|}{2} + 1 & \text{if } x < 1 \\ \dfrac{x}{2} + 1 & \text{if } 1 \leq x < 2 \\ -\dfrac{|x|}{2} + 1 & \text{if } 2 \leq x \end{cases}$ What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify yours answer. **[12 Marks]**

54. Show that the series $\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1.4}{3.6}\right)^2 + \ldots + \left(\dfrac{1.4.7\ldots\ldots(3n-2)}{3.6.9\ldots\ldots\ldots\ldots 3n}\right)^2 + \ldots\ldots$ converges **[15 Marks]**

55. Show that if $f : [a,b] \to R$ is a continuous function then $f([a,b]) = [c,d]$ form some real numbers c and d, $c \leq d$. **[15 Marks]**

56. Show that: $\lim\limits_{x \to 1} \sum\limits_{n=1}^{\infty} \dfrac{n^2 x^2}{n^4 + x^4} = \sum\limits_{n=1}^{\infty} \dfrac{n^2}{n^4 + 1}$ Justify all steps of your answer by quoting the theorems you are using **[15 Marks]**

57. Show that a bounded infinite subset R must have a limit point **[15 Marks]**

2008

58. (i) For $x > 0$, show $\dfrac{x}{1+x} < \log(1+x) < x$ **[6 Marks]**

(ii) Let $T = \left\{\dfrac{1}{n}, n \in N\right\} \cup \left\{1 + \dfrac{3}{2n}, n \in N\right\} \cup \left\{6 - \dfrac{1}{3n}, n \in N\right\}$. Find derived set T of T. Also find Supremum of T and greatest number of T. **[6 Marks]**

59. If $f : R \to R$ is continuous and $f(x+y) = f(x) + f(y),$ for all $x, y \in R$ then show that $f(x) = xf(1)$ for all $x \in R.$ **[12 Marks]**

60. Discuss the convergence of the series $\frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots.., x > 0.$ **[15 Marks]**

61. Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod_{2}^{\infty}\left(1 + \frac{1}{n^2 - 1}\right)$ **[15 Marks]**

62. Let f be a continuous function on $[0,1]$. Using first Mean Value theorem on Integration, prove that

$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0)$ **[15 Marks]**

63. (i) Prove that the sets $A = [0,1],\ B = (0,1)$ are equivalent sets. **[6 Marks]**

(ii) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x},\ x \in \left(0, \frac{\pi}{2}\right)$ **[9 Marks]**

2007

64. Show that the function given by $f(x,y) = \begin{cases} \frac{xy}{x^2 + 2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exists at $(0,0)$ **[12 Marks]**

65. Using Lagrange's mean value theorem, show that $|\cos b - \cos a| \le |b - a|$ **[12 Marks]**

66. Given a positive real number a and any natural number $n,$ prove that there exists one and only one positive real number ξ such that $\xi^n = a$ **[20 Marks]**

67. Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$ **[20 Marks]**

68. Rearrange the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ to converge to 1 **[20 Marks]**

2006

69. Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ **[12 Marks]**

70. Prove that the function f defined by $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ is nowhere continuous. **[12 Marks]**

71. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one value $\xi,\ a < \zeta < b$ for which $f''(\zeta) < 0.$ **[20 Marks]**

72. Show that the function given by $f(x,y) = \begin{cases} \frac{x^3 + 2y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$ (i) is continuous at $(0,0)$ (ii) possesses partial derivative $f_x(0,0)$ and $f_y(0,0)$ **[20 Marks]**

73. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ **[20 Marks]**

2005

74. If $u,\ v,\ w$ are the roots of the equation in λ and $\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1$, evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ **[12 Marks]**

75. Evaluate $\iiint \ln(x+y+z)\ dxdydz$ The integral being extended over all positive values of x,y,z such that $x+y+z\le 1$ **[12 Marks]**

76. If f' and g' exist for every $x\in[a,b]$ and if $g'(x)$ does not vanish anywhere (a,b), show that there exists c in (a,b) such that $\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$ **[30 Marks]**

77. Show that $\int_0^\infty e^{-t}t^{n-1}\ dt$ is an improper integral which converges for $n>0$ **[30 Marks]**

2004

78. Show that the function $f(x)$ defined as: $f(x)=\frac{1}{2^n},\ \frac{1}{2^{n+1}}\le x\le\frac{1}{2^n},\ n=0,1,2,.....$ and $f(0)=0$ is integrable in $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx=\frac{2}{3}$ **[12 Marks]**

79. Show that the function $f(x)$ defined on by: $f(x)=\begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational}\end{cases}$ is continuous only at $x=0$ **[12 Marks]**

80. If (x,y,z) be the lengths of perpendiculars drawn from any interior point P of triangle ABC on the sides BC,CA and AB respectively, then find the minimum value of $x^2+y^2+z^2$, the sides of the triangle ABC being a,b,c. **[20 Marks]**

81. Find the volume bounded by the paraboloid $x^2+y^2=az$, the cylinder $x^2+y^2=2ay$ and the plane $z=0$ **[20 Marks]**

82. Let $f(x)\ge g(x)$ for every x in $[a,b]$ and f and g are both bounded and Riemann integrable on $[a,b]$. At a point $c\in[a,b]$, let f and g be continuous and $f(c)>g(c)$ then prove that $\int_a^b f(x)\ dx>\int_a^b g(x)\ dx$ and hence show that $-\frac{1}{2}<\int_a^b\frac{x^3\cos 5x}{2+x^2}dx<\frac{1}{2}$ **[20 Marks]**

2003

83. Let a be a positive real number and $\{x_n\}$ sequence of rational numbers such that $\lim\limits_{n\to\infty}x_n=0$. Show that $\lim\limits_{n\to\infty}a^{x_n}=1$ **[12 Marks]**

84. If a continuous function of x satisfies the functional equation $f(x+y)=f(x)+f(y)$ then show that $f(x)=\alpha x$ where α is a constant. **[12 Marks]**

85. Show that the maximum value of $x^2y^2z^2$ subject to condition $x^2+y^2+z^2=c^2$ is $\frac{c^2}{27}$. Interpret the result

[20 Marks]

86. The axes of two equal cylinders intersect at right angles. If a be their radius, then find the volume common to the cylinder by the method of multiple integrals. **[20 Marks]**

87. Show that $\int_0^\infty \frac{dx}{1+x^2 sin^2 x}$ is divergent **[20 Marks]**

2002

88. Prove that the integral $\int_0^\infty x^{m-1}e^{-x}\ dx$ is convergent if and only if $m>0$. **[12 Marks]**

89. Find all the positive values of a for which the series $\sum_{n=1}^{\infty}\frac{(an)^n}{n!}$ converges. **[12 Marks]**

90. Test uniform convergence of the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^p}$, where $p>0$ **[20 Marks]**

91. Obtain the maxima and minima of $x^2+y^2+z^2-yz-zx-xy$ subject to condition $x^2+y^2+z^2-2x+2y+6z+9=0$ **[25 Marks]**

92. A solid hemisphere H of radius $'a'$ has density ρ depending on the distance R from the center of and is given by $\rho=k(2a-R)$ where k is a constant. Find the mass of the hemisphere by the method of multiple integrals

[15 Marks]

2001

93. Show that $\int_0^{\pi/2}\frac{x^n}{\sin^m x}dx$ exists if and only if $m<n+1$ **[12 Marks]**

94. If $\lim_{n\to\infty} a_n = l,$ then prove that $\lim_{n\to\infty}\frac{a_1+a_2+...a_n}{n}=l,$ **[12 Marks]**

95. A function f is defined in the interval (a,b) as follows

$$f(x)=\begin{cases}\frac{1}{q^2}\ \text{when } x=\frac{p}{q}\\ \frac{1}{q^3}\ \text{when } x=\sqrt{\frac{p}{q}}\end{cases}$$ where p,q relatively prime integers. $f(x)=0$ for all other values of x . Is f Riemann integrable? Justify your answer. **[20 Marks]**

96. Show that $U=xy+yz+zx$ has a maximum value when the three variables are connected by the relation $ax+by+cz=1$ and a,b,c are positive constants satisfying the condition $2(ab+bc+ca)>(a^2+b^2+c^2)$

[25 Marks]

97. Evaluate $\iiint\left(ax^2+by^2+cz^2\right)\ dxdydz$ taken throughout the region $x^2+y^2+z^2\le R^2$ **[15 Marks]**

2000

98. Given that the terms of a sequence $\{a_n\}$ are such that each $a_k,\ k\le 3,$ is the arithmetic mean of its two immediately preceding terms. Show that the sequence converges. Also find the limit of the sequence.

[12 Marks]

99. Determine the values of x for which the infinite product $\prod_{n=0}^{\infty}\left(1+\frac{1}{x^{2n}}\right)$ converges absolutely. Find its value whenever it converges. **[12 Marks]**

100. Suppose f is twice differentiable real valued function in $(0,\infty)$ and $M_{0,}M_1$ and M_2 the least upper bounds of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$ respectively in $(0,\infty)$. Prove for each $x>0,\ h>0$ that $f'(x)\frac{1}{2h}\left[f(x+2h)-f(x)\right]-hf'(u)$ for some $u\in(x,x+2h)$. Hence show that $\mathrm{M}_1{}^2\le 4\mathrm{M}_0\mathrm{M}_{2.}$ **[20 Marks]**

101. Evaluate $\iint_S\left(x^3dydz+x^2ydzdx+x^2zdxdy\right)$ by transforming into triple integral where S is the closed surface formed by the cylinder $x^2+y^2=a^2, 0\le z\le b$ and the circular disc $x^2+y^2\le a^2, z=0$ and $x^2+y^2\le a^2, z=b$ **[20 Marks]**

1999

102. Let A be a subset of the metric space (M,ρ). If (A,ρ) is compact, then show that A is a closed subset of (M,ρ) **[20 Marks]**

103. A sequence $\{S_n\}$ is defined by the recursion formula $\mathrm{S}_{n+1}=\sqrt{3S_n}, S_1=1$. Does this sequence converge? If so, find $\lim S_n$ **[20 Marks]**

104. Test for convergence the integral $\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx$ **[20 Marks]**

105. Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2=225,\ z=0$ **[20 Marks]**

106. Show that the double integral $\iint_R\frac{x-y}{(x+y)^3}\,dxdy$ does not exist over $R=[0,1;0,1]$ **[20 Marks]**

107. Verify the Gauss divergence theorem for $\overline{F}=4x\hat{e}_x-2y^2\hat{e}_y+z^2\hat{e}_z$ taken over the region bounded by $x^2+y^2=4,\ z=0$ and $z=3$ where $\hat{e}_{x,}\hat{e}_{y,}\hat{e}_z$ are unit vectors along $x-, y-$ and $z-$ directions respectively. **[20 Marks]**

1998

108. Let X be a metric space and $E\subset X$. Show that
(i) Interior of E is the largest open set contained in E
(ii) Boundary of E =(closure of E) $\bigcap$ (closure of $X-E$) **[20 Marks]**

109. Let (X,d) and (Y,e) be metric spaces with X compact and $f: X\to Y$ continuous. Show that f is uniformly continuous. **[20 Marks]**

110. Show that the function $f(x,y)=2x^4-3x^2y+y^2$ has $(0,0)$ as the only critical point but the function neither has a minima nor maxima at $(0,0)$ **[20 Marks]**

111. Test the convergence of the integral $\int_0^{\infty}e^{-ax}\frac{\sin x}{x}\,dx,\ \ a\ge 0$ **[20 Marks]**

112. Test the series $\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$ for uniform convergence. **[20 Marks]**

113. Let $f(x)=x$ and $g(x)=x^2$. Does $\int_0^1 fog$ exist? If it exists then find its value **[20 Marks]**

1997

114. Show that a non-empty set P in R^n each of whose points is a limit-point is uncountable. **[20 Marks]**

115. Show that $\iiint\limits_D xyz\ dxdydz = \frac{a^2b^2c^2}{6}$ where domain D is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ **[20 Marks]**

116. If $u=\sin^{-1}\left[\left(x^2+y^2\right)^{1/5}\right]$, Prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\tan u(2\tan^2 u-3)$ **[20 Marks]**

1996

117. Let F be the set of all real valued bounded continuous functions defined on the closed interval $[0,1]$. Let d be a mapping of $F\times F$ into $R,$ the set of real numbers, defined by $d(f,g)=\int_0^1 |f(x)-g(x)|\ dx\quad \forall f,g\in F$.
Verify that d is a metric for F **[20 Marks]**

118. Prove that a compact set in a metric space is a closed set. **[20 Marks]**

119. Let $C[a,b]$ denote the set of all functions f on $[a,b]$ which have continuous derivatives at all points of $I=[a,b]$ .For $f,g\in C[a,b]$ define $d(f,g)=|f(a)-g(b)|+\sup\{|f'(x)-g'(x)|,x\in I\}$. Show that the space $(C[a,b],d)$ is a complete. **[20 Marks]**

120. A function f is defined in the interval (a,b) as follows:
$$f(x)=\begin{cases} q^{-2} & \text{when } x=pq^{-1} \\ q^{-3} & \text{when } x=\left(pq^{-1}\right)^{1/2}\end{cases}$$
where p,q are relatively prime integers; $f(x)=0,$ for all other values of x. Is f Riemann integrable? Justify your answer. **[20 Marks]**

121. Test for uniform convergence, the series $\sum_{n=1}^{\infty}\frac{2^n x^{2n-1}}{1+x^{2n}}$ **[20 Marks]**

122. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2}\sin x\sin^{-1}\left(\sin x\sin y\right)\ dxdy$ **[20 Marks]**

1995

123. Let K and F be nonempty disjoint closed subjects of R^2. If K is bounded, show that there exists $\delta>0$ such that $d(x,y)\ge\delta$ for $x\in K$ and $y\in F$ where $d(x,y)$ is the usual distance between x and y. **[20 Marks]**

124. Let f be a continuous real function on R such that f maps open interval into open intervals. Prove that f is monotonic. **[20 Marks]**

125. Let $c_n\ge 0$ for all positive integers n such that is convergent. Suppose $\{S_n\}$ is a sequence of distinct points in (a,b) For $x\in[a,b]$, define $\alpha(x)=\sum c_n\ \{n:x>S_n\}$. Prove that α is an increasing function. If f a continuous real function on $[a,b],$ show that $\int_a^b fd\alpha=\sum c_n f(S_n)$ **[20 Marks]**

126. Suppose f maps an open ball $U \subset R^n$ into R^m and f is differentiable on U. Suppose there exists a real number $M > 0$ such that $\|f(x)\| \le M \ \forall x \in U$. Prove that $|f(b) - f(a)| \le M|b-a| \ \forall a, b \in U$ **[20 Marks]**

127. Find and classify the extreme values of the function $f(x,y) = x^2 + y^2 + x + y + xy$ **[20 Marks]**

128. Suppose α is real number not equals to $n\pi$ for any integer n. Prove that

$$\int_0^\infty \int_0^\infty e^{-(x^2+2xy\cos\alpha+y^2)} \ dxdy = \frac{\alpha}{2\sin\alpha}$$ **[20 Marks]**

1994

129. Examine the (i) absolute convergence (ii) uniform convergence of the series $(1-x) + x(1-x) + x^2(1-x) + ...$ in $[-c,1],\ 0 < c < 1$ **[20 Marks]**

130. Prove that $S(x) = \sum \frac{1}{n^p + n^q x^2}$, $p > 1$ is uniformly convergent for all values of x and can be differentiate term by term if $q < 3p < 2$ **[20 Marks]**

131. Let the function f be defined on $[0,1]$ by the condition $f(x) = 2rx$ when $\frac{1}{r+1} < x < \frac{1}{r},\ r > 0$ Show that f is Riemann integrable in $[0,1]$ and $\int_0^1 f(x)\ dx = \frac{\pi^2}{6}$ **[20 Marks]**

132. By means of substitution $x+y+z = u, y+z = uv, z = uvw$ evaluate $\iiint (x+y+z)^n xyz \ dxdydz$ taken over the volume bounded by $x = 0, y = 0, z = 0, x+y+z = 1$ **[20 Marks]**

1993

133. Examine for Riemann integrability over $[0,2]$ of the function defined in $[0,2]$ by

$$f(x) = \begin{cases} x + x^2, & \text{for rational valuesof } x \\ x^2 + x^3, & \text{for irrational values of } x \end{cases}$$ **[20 Marks]**

134. Prove that $\int_0^\infty \frac{\sin x}{x}\ dx$ converges and conditionally converges. **[20 Marks]**

135. Evaluate $\iiint \frac{dxdydz}{x+y+z+1}$ over the volume bounded by the coordinate planes and the plane $x+y+z=1$

[20 Marks]

1992

136. If we metrize the space of functions continuous on $[a,b]$ by taking $p(x,y) = \sqrt{\int_a^b [x(t) - y(t)]^2 \ dt}$ then show that the resulting metric space is NOT complete **[20 Marks]**

137. Examine $2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y - 4z$ for extreme values **[20 Marks]**

138. If $U_n = \dfrac{1+nx}{ne^{nx}} - \dfrac{1+(n+1)x}{(n+1)e^{(n+1)x}}$, $0<x<1$ Prove that $\dfrac{d}{dx}\sum U_n = \sum \dfrac{d}{dx}U_n$ Is the series uniformly convergent in $(0,1)$? Justify your claim. **[20 Marks]**

139. Find the upper and lower Riemann integral for the function defined in the interval $[0,1]$ as follows

$$\begin{cases} \sqrt{1-x^2} & \text{when } x \text{ is rational} \\ 1-x & \text{when } x \text{ is irrational} \end{cases}$$

and show that is NOT Riemann integrable in $[0,1]$. **[20 Marks]**

140. Discuss the convergence or divergence of $\int_0^\infty \dfrac{x^\beta}{1+x\alpha\sin^2 x}dx,\ \alpha>\beta>0$ **[20 Marks]**

141. Evaluate $\iint \sqrt{\dfrac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}\,dxdy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ **[20 Marks]**