RAMANASRI
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Calculus

Indian Forest Service (IFoS) Maths Optional Previous Year Questions (PYQ) from 2020 to 2009

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IAS, UPSC, CIVIL SERVICES, IFoS MAINS EXAMINATION MATHEMATICS OPTIONAL STUDY MATERIALS

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2020

1. Give that $f(x+y)=f(x)f(y)$, $f(0)\neq 0$ for all real x, y and $f'(0)=2$ show that for all x, $f'(x)=2f(x)$. Hence find $f(x)$ **[8 Marks]**

2. Find the Taylors series expansion for the function $f(x)=\log(1+x)-1\leq x\leq\infty$ about $x=2$ with Lagrange's from of remainder after 3-terms. **[8 Marks]**

3. Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube. **[15 Marks]**

4. Find the asymptotes of the curve $x^3+3x^2y-4y-x+y+3=0$ **[10 Marks]**

5. Evaluate: (i) $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$ (ii) Evaluate the following integral: $\int_{-\infty}^{\infty} xe^{-x^2}dx$ **[6+9=15 Marks]**

2019

6. Justify by using Rolle's Theorem or mean values theorem that there is no number k for which the equation $x^3-3x+k=0$ has two distinct solutions in the interval $[-1, 1]$. **[8 Marks]**

7. Evaluate $\int_{(0,\,0)}^{(2,\,1)}(10x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$. **[8 Marks]**

8. Find the centroid of the solid generated by revolving the upper. Half of the cardioids $r=a(1+\cos\theta)$ bounded by the line $\theta=0$ about the initial line. Take the density of the solid as uniform. **[10 Marks]**

9. The dimensions of a rectangular box are linear functions of time $l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and the height is decreasing at the rate 3 cm/ sec, find the rates at which the volume V and the surface area S are changing with respect to time. If $l(0)=10$, $w(0)=8$ and $h(0)=20$, is V increasing or decreasing, when $t=5$ sec? What about S, when $t=5$ sec? **[10 Marks]**

10. Determine the extreme values of the function $f(x,y)=3x^2-6x+2y^2-4y$ in the region $\{(x,\ y)\in R^2 : 3x^2+2y^2\leq 20\}$. **[10 Marks]**

2018

11. Show that the maximum rectangle inscribed in a circle is a square. **[8 Marks]**

12. If $f:[a,b]\to R$ be continuous in $[a,b]$ and derivable in (a,b) where $0\leq a\leq b$ show that for $c\in(a,b)$

$$f(b)-f(a)=cf'(c)\log(b/a)$$ **[8 Marks]**

13. If ϕ and ψ be two functions derivable in $\left[a,b\right]$ and $\phi(x)\,\psi'(x)-\psi(x)\,\phi'(x)>0$ for any x in this interval, then show that between two consecutive roots of, $\phi(x)=0$ in $\left[a,b\right]$ there lies exactly one root of $\psi(x)=0$ **[10 Marks]**

14. If $f=f(u,v)$ where $u=e^{x}\cos y$ and $v=e^{x}\sin,y$, show that $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$

[10 Marks]

15. Evaluate $\iint_R (x^2+xy)dxdy$ over the region R bounded by $xy=1, y=0$, $y=x$ and $x=2$

[10 Marks]

16. Show that the functions $u=x+y+z$,$v=xy+yz+zx$ and $w=x^3+y^3+z^3-3xyz$ are dependent and find the relation between them. **[10 Marks]**

2017

17. Let $u(x,y)=ax^2+2hxy+by^2$ and $v(x,y)=Ax^2+2Hxy+By^2$ Find the Jacobian $J=\frac{\partial(u,v)}{\partial(x,y)}$ and hence show that u,v are independent unless $\frac{a}{A}=\frac{b}{B}=\frac{h}{H}$ **[8 Marks]**

18. Show that

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p,q>-1$$

Hence evaluate the following integrals:

(i) $\int_0^{\pi/2}\sin^4 x\cos^5 x dx$

(ii) $\int_0^1 x^3(1-x^2)^{5/2}dx$

(iii) $\int_0^1 x^4(1-x)^3 dx$ **[10 Marks]**

19. Find the maxima and minima for the function

$$f(x,y)=x^3+y^3-3x-12y+20$$

Also find the saddle points (if any) for the function. **[10 Marks]**

20. Evaluate the integral $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy,$ by changing to polar coordinate Hence show that

$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$ **[10 Marks]**

21. A function $f(x,y)$ is defined as follows:

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & \text{if } (x,y)\neq 0\\ 0, & \text{if } (x,y)=0\end{cases}$$

Show that $f_{xy}(0,0)=f_{yx}(0,0)$. **[10 Marks]**

2016

22. Show that $\dfrac{1}{(1+x)}<\log(1+x)<x$ for $x>0$ **[8 Marks]**

23. Examine if the function $f(x,y)=\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0)$ and $f(0,0)=,0$ is continuous at $(0,0)$ Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at point other than origin. **[8 Marks]**

24. If the point $(2,3)$ is the mid-point of a chord of the parabola $y^2=4x$ then obtain of the chord. **[8 Marks]**

25. After changing the order of integration of $\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy$, show that $\int_0^\infty \dfrac{\sin nx}{x}dx=\dfrac{\pi}{2}$ **[10 Marks]**

26. Using mean value theorem, find a point on the curve $y=\sqrt{x-2}$ defined on $[2,3]$ where the tangent is parallel to the chord joining the end points of the curve. **[10 Marks]**

27. Using Lagrange's method of multipliers, find the point on the plane $2x+3y+4z=5$ which is closest to the point $(1,0,0)$ **[10 Marks]**

28. Obtain the area between the curve $r=3(\sec\theta+\cos\theta)$ and its asymptote $x=3$ **[10 Marks]**

29. Show that the integral $\int_0^\infty e^{-x}x^{\alpha-1}dx,\ \alpha>0$ exists, by separately taking the cases for $\alpha\geq 1$ and $0<\alpha<1$ **[10 Marks]**

30. Prove that $\Gamma(2z)=\dfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\dfrac{1}{2}\right)$. **[10 Marks]**

2015

31. Let $f(x)$ be a real –valued function defined on the interval $(-5,\ 5)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$ for all $x\in(-5,\ 5)$. Let $f^{-1}(x)$ be the interval function of $f(x)$. Find $(f^{-1})'(2)$ **[8 Marks]**

32. For $x>0$, let $f(x)=\int_1^x \dfrac{1nt}{1+t}dt$. Evaluate $f(e)+f\left(\dfrac{1}{e}\right)$ **[8 Marks]**

33. A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensional of the box so that the total surface is a minimum. **[10 Marks]**

34. Find the area enclosed by the curve in which the plane $z=2$ cuts the ellipsoid

$$\frac{x^2}{25}+y^2+\frac{z^2}{5}=1$$ **[10 Marks]**

35. Consider the three-dimensional region R bounded by $x+y+z=1, y=0, z=0$. Evaluate

$$\iiint_R (x^2+y^2+z^2)\ dxdydz$$ **[10 Marks]**

36. Evaluate $\lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x}-\frac{3}{x^4}\right)$ **[10 Marks]**

37. The forces P, Q and R act along three straight line $y=b, z=-c, z=c, x=-a$ and $x=a, y=-b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action. **[10 Marks]**

2014

38. Show that function given by

$$f(x)=\begin{cases}\frac{x(e^{1/x}-1)}{(e^{1/x}+1)}, & x\neq 0\\ 0, & x=0\end{cases}$$ is continuous but not differentiable at $x=0$ **[8 Marks]**

39. Evaluate $\iint_R y\frac{\sin x}{x}dxdy$ over R where $R=\left\{(x,y): y\le x\le \frac{\pi}{2}, 0\le y\le \frac{\pi}{2}\right\}$ **[8 Marks]**

40. If $xyz=a^3$ then show that the minimum value of $x^2+y^2+z^2$ is $3a^2$ **[10 Marks]**

41. Evaluate the integral $I=\int_0^\infty 2^{-ax^2}dx$ using Gamma function. **[10 Marks]**

42. Let f be a real valued function defined on $[0,1]$ as follows:

$$f(x)=\begin{cases}\frac{1}{a^{r-1}}, & \frac{1}{a^r}<x\le\frac{1}{a^{r-1}}, r=1,2,3......\\ 0 & x=0\end{cases}$$ where a is integer greater than 2. Show that

$\int_0^1 f(x)dx$ exists and is equal to $\frac{a}{a+1}$ **[10 Marks]**

43. Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}dxdy$ over the region R bounded between $0\le x\le\frac{y^2}{2}$ and $0\le y\le 2$ **[10 Marks]**

44. A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$ radius of hemisphere. **[15 Marks]**

2013

45. Evaluate the integral $\int_0^\infty \int_0^x xe^{-x^2/y} dydx$ by changing the order of integration.

[8 Marks]

46. Find C of the Mean value theorem, If $f(x) = x(x-1)(x-2), a = 0, b = \frac{1}{2}$ and C has usual meaning

[8 Marks]

47. Prove that if $a_0, a_1, a_2, ..., a_n$ are the real number such that

$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + ... + \frac{a_{n-1}}{2} + a_n = 0$ then there exists at least one real number x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n = 0$ **[10 Marks]**

48. Evaluate $\int_0^{\pi/2} \frac{x \sin x \cos dx}{\sin^4 x \cos^4 x}$ **[10 Marks]**

49. Find all the asymptotes of the curve $x^4 - y^4 + 3x^2 y + 3xy^3 + xy = 0$ **[10 Marks]**

2012

50. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$ show that, $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$ **[8 Marks]**

51. If $u = Ae^{-gx} \sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$ **[8 Marks]**

52. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq cm. **[10 Marks]**

53. Show that the function defined as $f(x) = \begin{cases} \frac{\sin 2x}{x} \text{ when } x \neq 0 \\ 1 \quad \text{wnen } x = 0 \end{cases}$ has removable discontinuity at the origin. **[10 Marks]**

54. Find by triple integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$. **[10 Marks]**

55. Evaluate the following in terms of Gamma function $\int_0^a \sqrt{\left(\frac{x^3}{a^3 - x^3}\right)} dx$ **[10 Marks]**

2011

56. Show that the function defined by $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$ is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat. **[10 Marks]**

57. Let the function f be defined by $f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } 0 \leq t \leq 1 \\ 4, & \text{for } t > 1 \end{cases}$

(i) Determine the function $F(x) = \int_0^x f(t)dt.$

(ii) Where is F non-differentiable? Justify your answer. **[10 Marks]**

58. Show that the equation $3^x + 4^x = 5^x$ has exactly one root. **[8 Marks]**

59. Test for convergence the integral $\int_0^\infty \sqrt{xe^{-x}}\,dx.$ **[8 Marks]**

60. Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$ **[12 Marks]**

61. Show that function defined by $f(x,y,z) = 3\log(x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3, (x,y,z) \neq (0,0,0)$ has only one extreme Value, $\log\left(\dfrac{3}{e^2}\right)$ **[12 Marks]**

2010

62. Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0.$ **[8 Marks]**

63. Let f be a function defined on R such that $f(x+y) = f(x) + f(y), x, y \in R.$ if f is differentiable at one point of R, then prove that f is differentiable on R **[8 Marks]**

64. Discuss the convergence of the integral $\int_0^\infty \dfrac{dx}{1 + x^4 \sin^2 x}$ **[10 Marks]**

65. Find the extreme value of xyz if $x + y + z = a.$ **[10 Marks]**

66. Let $f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

Show that:

(i) $f_{xy}(0,0) \neq f_{yx}(0,0)$

(ii) f is differentiable at $(0,0)$ **[10 Marks]**

67. Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$

[10 Marks]

2009

68. (i) Find the difference between in the maximum and the minimum of the function $\left(a-\frac{1}{a}-x\right)(4-3x^2)$ where a is a constant and greater than zero. **[5 Marks]**

(ii) If $f(h)=f(0)+hf'(0)+\frac{h^2}{2!}+f''(\theta h),\ 0<\theta<1$. Find θ, when $h=1$ and $f(x)=(1-x)^{5/2}5+5=10$ **[5 Marks]**

69. Evaluate: (i) $\int_0^{\pi/2} \frac{\sin^2 x dx}{\sin x+\cos x}$ **[6 Marks]** (ii) $\int_1^{\infty} \frac{x^2 dx}{(1+x^2)^2}$ **[4 Marks]**

70. The adiabatic law for the expansion of air is $PV^{1\cdot 4}=K$, where K is a constant. If at a given time the Volume is observed to be 50 c.c. and the pressure in 30 kg per square centimeter at what rate is the pressure changing if the volume is decreasing at the rate of 2 c.c. per second? **[10 Marks]**

71. Determine the asymptotes of the curve $x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$. **[10 Marks]**

72. Evaluate: $\iint_D x\sin(x+y)dxdy$. where D is the region bounded by $0\le x\le \pi$ and $0\le y\le \frac{\pi}{2}$

[10 Marks]

73. Evaluate $\iiint (x+y+z+1)^4 dxdydz$ over the region defined by $x\ge 0, y\ge 0, z\ge 0$. and $x+y+z\le 1$

[10 Marks]